

## Proof of Theorem 261ij

The theorem to be proved is

$$Qx = 2 \uparrow \text{Length } x \rightarrow Q(x \oplus \underline{0}) = 2 \uparrow \text{Length}(x \oplus \underline{0}) \quad \& \quad Q(x \oplus \underline{1}) = 2 \uparrow \text{Length}(x \oplus \underline{1})$$

Suppose the theorem does not hold. Then, with the variables held fixed,

$$(H) \quad [[(Qx) = (2 \uparrow (\text{Length } x))] \quad \& \quad [\neg (Q(x \oplus \underline{0})) = (2 \uparrow (\text{Length}(x \oplus \underline{0}))) \quad \vee \quad \neg (Q(x \oplus \underline{1})) = (2 \uparrow (\text{Length}(x \oplus \underline{1})))]]$$

### Special cases of the hypothesis and previous results:

- 0:  $2 \uparrow (\text{Length } x) = Qx$  from H: $x$
- 1:  $\neg 2 \uparrow (\text{Length}(x \oplus \underline{0})) = Q(x \oplus \underline{0}) \quad \vee \quad \neg 2 \uparrow (\text{Length}(x \oplus \underline{1})) = Q(x \oplus \underline{1})$  from H: $x$
- 2:  $\text{Length}(x \oplus \underline{0}) = S(\text{Length } x)$  from [259](#); $x$
- 3:  $\text{Length}(x \oplus \underline{1}) = S(\text{Length } x)$  from [259](#); $x$
- 4:  $2 \cdot (2 \uparrow (\text{Length } x)) = 2 \uparrow (S(\text{Length } x))$  from [126](#); $2$ ; $\text{Length } x$
- 5:  $(Qx) \cdot (Q\underline{0}) = Q(x \oplus \underline{0})$  from [180](#); $x$ ; $\underline{0}$
- 6:  $(Qx) \cdot (Q\underline{1}) = Q(x \oplus \underline{1})$  from [180](#); $x$ ; $\underline{1}$
- 7:  $Q\underline{0} = 2$  from [191](#)
- 8:  $Q\underline{1} = 2$  from [192](#)
- 9:  $(Qx) \cdot 2 = 2 \cdot (Qx)$  from [105](#); $Qx$ ; $2$

### Equality substitutions:

- 10:  $\neg 2 \uparrow (\text{Length } x) = Qx \quad \vee \quad \neg 2 \cdot (2 \uparrow (\text{Length } x)) = 2 \uparrow (S(\text{Length } x)) \quad \vee \quad 2 \cdot (Qx) = 2 \uparrow (S(\text{Length } x))$
- 11:  $\neg \text{Length}(x \oplus \underline{0}) = S(\text{Length } x) \quad \vee \quad 2 \uparrow (\text{Length}(x \oplus \underline{0})) = 2 \cdot (Qx) \quad \vee \quad \neg 2 \uparrow (S(\text{Length } x)) = 2 \cdot (Qx)$
- 12:  $\neg \text{Length}(x \oplus \underline{1}) = S(\text{Length } x) \quad \vee \quad 2 \uparrow (\text{Length}(x \oplus \underline{1})) = 2 \cdot (Qx) \quad \vee \quad \neg 2 \uparrow (S(\text{Length } x)) = 2 \cdot (Qx)$
- 13:  $\neg Q\underline{0} = 2 \quad \vee \quad \neg (Qx) \cdot (Q\underline{0}) = Q(x \oplus \underline{0}) \quad \vee \quad (Qx) \cdot (2) = Q(x \oplus \underline{0})$
- 14:  $\neg Q\underline{1} = 2 \quad \vee \quad \neg (Qx) \cdot (Q\underline{1}) = Q(x \oplus \underline{1}) \quad \vee \quad (Qx) \cdot (2) = Q(x \oplus \underline{1})$
- 15:  $\neg (Qx) \cdot 2 = 2 \cdot (Qx) \quad \vee \quad \neg Q(x \oplus \underline{0}) = (Qx) \cdot 2 \quad \vee \quad Q(x \oplus \underline{0}) = 2 \cdot (Qx)$

$$16: \neg(Qx) \cdot 2 = 2 \cdot (Qx) \quad \vee \quad \neg Q(x \oplus \underline{1}) = (Qx) \cdot 2 \quad \vee \quad Q(x \oplus \underline{1}) = 2 \cdot (Qx)$$

$$17: \neg Q(x \oplus \underline{0}) = 2 \cdot (Qx) \quad \vee \quad 2 \uparrow(\text{Length}(x \oplus \underline{0})) = Q(x \oplus \underline{0}) \quad \vee \quad \neg 2 \uparrow(\text{Length}(x \oplus \underline{0})) = 2 \cdot (Qx)$$

$$18: \neg Q(x \oplus \underline{1}) = 2 \cdot (Qx) \quad \vee \quad 2 \uparrow(\text{Length}(x \oplus \underline{1})) = Q(x \oplus \underline{1}) \quad \vee \quad \neg 2 \uparrow(\text{Length}(x \oplus \underline{1})) = 2 \cdot (Qx)$$

### Inferences:

$$19: \neg 2 \cdot (2 \uparrow(\text{Length}x)) = 2 \uparrow(\text{S}(\text{Length}x)) \quad \vee \quad 2 \uparrow(\text{S}(\text{Length}x)) = 2 \cdot (Qx) \quad \text{by}$$

$$0: 2 \uparrow(\text{Length}x) = Qx$$

$$10: \neg 2 \uparrow(\text{Length}x) = Qx \quad \vee \quad \neg 2 \cdot (2 \uparrow(\text{Length}x)) = 2 \uparrow(\text{S}(\text{Length}x)) \quad \vee$$

$$2 \uparrow(\text{S}(\text{Length}x)) = 2 \cdot (Qx)$$

$$20: 2 \uparrow(\text{Length}(x \oplus \underline{0})) = 2 \cdot (Qx) \quad \vee \quad \neg 2 \uparrow(\text{S}(\text{Length}x)) = 2 \cdot (Qx) \quad \text{by}$$

$$2: \text{Length}(x \oplus \underline{0}) = \text{S}(\text{Length}x)$$

$$11: \neg \text{Length}(x \oplus \underline{0}) = \text{S}(\text{Length}x) \quad \vee \quad 2 \uparrow(\text{Length}(x \oplus \underline{0})) = 2 \cdot (Qx) \quad \vee$$

$$\neg 2 \uparrow(\text{S}(\text{Length}x)) = 2 \cdot (Qx)$$

$$21: 2 \uparrow(\text{Length}(x \oplus \underline{1})) = 2 \cdot (Qx) \quad \vee \quad \neg 2 \uparrow(\text{S}(\text{Length}x)) = 2 \cdot (Qx) \quad \text{by}$$

$$3: \text{Length}(x \oplus \underline{1}) = \text{S}(\text{Length}x)$$

$$12: \neg \text{Length}(x \oplus \underline{1}) = \text{S}(\text{Length}x) \quad \vee \quad 2 \uparrow(\text{Length}(x \oplus \underline{1})) = 2 \cdot (Qx) \quad \vee$$

$$\neg 2 \uparrow(\text{S}(\text{Length}x)) = 2 \cdot (Qx)$$

$$22: 2 \uparrow(\text{S}(\text{Length}x)) = 2 \cdot (Qx) \quad \text{by}$$

$$4: 2 \cdot (2 \uparrow(\text{Length}x)) = 2 \uparrow(\text{S}(\text{Length}x))$$

$$19: \neg 2 \cdot (2 \uparrow(\text{Length}x)) = 2 \uparrow(\text{S}(\text{Length}x)) \quad \vee \quad 2 \uparrow(\text{S}(\text{Length}x)) = 2 \cdot (Qx)$$

$$23: \neg Q\underline{0} = 2 \quad \vee \quad Q(x \oplus \underline{0}) = (Qx) \cdot 2 \quad \text{by}$$

$$5: (Qx) \cdot (Q\underline{0}) = Q(x \oplus \underline{0})$$

$$13: \neg Q\underline{0} = 2 \quad \vee \quad \neg (Qx) \cdot (Q\underline{0}) = Q(x \oplus \underline{0}) \quad \vee \quad Q(x \oplus \underline{0}) = (Qx) \cdot 2$$

$$24: \neg Q\underline{1} = 2 \quad \vee \quad Q(x \oplus \underline{1}) = (Qx) \cdot 2 \quad \text{by}$$

$$6: (Qx) \cdot (Q\underline{1}) = Q(x \oplus \underline{1})$$

$$14: \neg Q\underline{1} = 2 \quad \vee \quad \neg (Qx) \cdot (Q\underline{1}) = Q(x \oplus \underline{1}) \quad \vee \quad Q(x \oplus \underline{1}) = (Qx) \cdot 2$$

$$25: Q(x \oplus \underline{0}) = (Qx) \cdot 2 \quad \text{by}$$

$$7: Q\underline{0} = 2$$

$$23: \neg Q\underline{0} = 2 \quad \vee \quad Q(x \oplus \underline{0}) = (Qx) \cdot 2$$

- 26:  $Q(x \oplus \underline{1}) = (Qx) \cdot 2$  by  
8:  $Q\underline{1} = 2$   
24:  $\neg Q\underline{1} = 2 \vee Q(x \oplus \underline{1}) = (Qx) \cdot 2$
- 27:  $\neg Q(x \oplus \underline{0}) = (Qx) \cdot 2 \vee Q(x \oplus \underline{0}) = 2 \cdot (Qx)$  by  
9:  $(Qx) \cdot 2 = 2 \cdot (Qx)$   
15:  $\neg (Qx) \cdot 2 = 2 \cdot (Qx) \vee \neg Q(x \oplus \underline{0}) = (Qx) \cdot 2 \vee Q(x \oplus \underline{0}) = 2 \cdot (Qx)$
- 28:  $\neg Q(x \oplus \underline{1}) = (Qx) \cdot 2 \vee Q(x \oplus \underline{1}) = 2 \cdot (Qx)$  by  
9:  $(Qx) \cdot 2 = 2 \cdot (Qx)$   
16:  $\neg (Qx) \cdot 2 = 2 \cdot (Qx) \vee \neg Q(x \oplus \underline{1}) = (Qx) \cdot 2 \vee Q(x \oplus \underline{1}) = 2 \cdot (Qx)$
- 29:  $2 \uparrow (\text{Length}(x \oplus \underline{0})) = 2 \cdot (Qx)$  by  
22:  $2 \uparrow (\text{S}(\text{Length}x)) = 2 \cdot (Qx)$   
20:  $2 \uparrow (\text{Length}(x \oplus \underline{0})) = 2 \cdot (Qx) \vee \neg 2 \uparrow (\text{S}(\text{Length}x)) = 2 \cdot (Qx)$
- 30:  $2 \uparrow (\text{Length}(x \oplus \underline{1})) = 2 \cdot (Qx)$  by  
22:  $2 \uparrow (\text{S}(\text{Length}x)) = 2 \cdot (Qx)$   
21:  $2 \uparrow (\text{Length}(x \oplus \underline{1})) = 2 \cdot (Qx) \vee \neg 2 \uparrow (\text{S}(\text{Length}x)) = 2 \cdot (Qx)$
- 31:  $Q(x \oplus \underline{0}) = 2 \cdot (Qx)$  by  
25:  $Q(x \oplus \underline{0}) = (Qx) \cdot 2$   
27:  $\neg Q(x \oplus \underline{0}) = (Qx) \cdot 2 \vee Q(x \oplus \underline{0}) = 2 \cdot (Qx)$
- 32:  $Q(x \oplus \underline{1}) = 2 \cdot (Qx)$  by  
26:  $Q(x \oplus \underline{1}) = (Qx) \cdot 2$   
28:  $\neg Q(x \oplus \underline{1}) = (Qx) \cdot 2 \vee Q(x \oplus \underline{1}) = 2 \cdot (Qx)$
- 33:  $\neg Q(x \oplus \underline{0}) = 2 \cdot (Qx) \vee 2 \uparrow (\text{Length}(x \oplus \underline{0})) = Q(x \oplus \underline{0})$  by  
29:  $2 \uparrow (\text{Length}(x \oplus \underline{0})) = 2 \cdot (Qx)$   
17:  $\neg Q(x \oplus \underline{0}) = 2 \cdot (Qx) \vee 2 \uparrow (\text{Length}(x \oplus \underline{0})) = Q(x \oplus \underline{0}) \vee \neg 2 \uparrow (\text{Length}(x \oplus \underline{0})) = 2 \cdot (Qx)$
- 34:  $\neg Q(x \oplus \underline{1}) = 2 \cdot (Qx) \vee 2 \uparrow (\text{Length}(x \oplus \underline{1})) = Q(x \oplus \underline{1})$  by  
30:  $2 \uparrow (\text{Length}(x \oplus \underline{1})) = 2 \cdot (Qx)$   
18:  $\neg Q(x \oplus \underline{1}) = 2 \cdot (Qx) \vee 2 \uparrow (\text{Length}(x \oplus \underline{1})) = Q(x \oplus \underline{1}) \vee \neg 2 \uparrow (\text{Length}(x \oplus \underline{1})) = 2 \cdot (Qx)$
- 35:  $2 \uparrow (\text{Length}(x \oplus \underline{0})) = Q(x \oplus \underline{0})$  by  
31:  $Q(x \oplus \underline{0}) = 2 \cdot (Qx)$   
33:  $\neg Q(x \oplus \underline{0}) = 2 \cdot (Qx) \vee 2 \uparrow (\text{Length}(x \oplus \underline{0})) = Q(x \oplus \underline{0})$

36:  $2 \uparrow (\text{Length}(x \oplus \underline{1})) = \mathbf{Q}(x \oplus \underline{1})$  by

$$32: \mathbf{Q}(x \oplus \underline{1}) = 2 \cdot (\mathbf{Q}x)$$

$$34: \neg \mathbf{Q}(x \oplus \underline{1}) = 2 \cdot (\mathbf{Q}x) \quad \vee \quad 2 \uparrow (\text{Length}(x \oplus \underline{1})) = \mathbf{Q}(x \oplus \underline{1})$$

37:  $\neg 2 \uparrow (\text{Length}(x \oplus \underline{1})) = \mathbf{Q}(x \oplus \underline{1})$  by

$$35: 2 \uparrow (\text{Length}(x \oplus \underline{0})) = \mathbf{Q}(x \oplus \underline{0})$$

$$1: \neg 2 \uparrow (\text{Length}(x \oplus \underline{0})) = \mathbf{Q}(x \oplus \underline{0}) \quad \vee \quad \neg 2 \uparrow (\text{Length}(x \oplus \underline{1})) = \mathbf{Q}(x \oplus \underline{1})$$

38: *QEA* by

$$36: 2 \uparrow (\text{Length}(x \oplus \underline{1})) = \mathbf{Q}(x \oplus \underline{1})$$

$$37: \neg 2 \uparrow (\text{Length}(x \oplus \underline{1})) = \mathbf{Q}(x \oplus \underline{1})$$