Proof of Theorem 260ij

The theorem to be proved is

 $\text{Length}(x \oplus y) = \text{Length} \, x + \text{Length} \, y \quad \rightarrow \quad \text{Length} \left(x \oplus (y \oplus \underline{0}) \right) = \text{Length} \, x + \text{Length}(y \oplus \underline{0}) \\ \& \quad \text{Length} \left(x \oplus (y \oplus \underline{1}) \right) = \text{Length} \, x + \text{Length}(y \oplus \underline{1})$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(\text{Length}(x \oplus y)) = ((\text{Length}x) + (\text{Length}y))] \& [\neg (\text{Length}(x \oplus (y \oplus \underline{0}))) = ((\text{Length}x) + (\text{Length}(y \oplus \underline{0}))) \lor \neg (\text{Length}(x \oplus (y \oplus \underline{1}))) = ((\text{Length}x) + (\text{Length}(y \oplus \underline{1})))]]$

Special cases of the hypothesis and previous results:

0: $(\text{Length}x) + (\text{Length}y) = \text{Length}(x \oplus y)$ from H:x:y

1: \neg (Lengthx) + (Length $(y \oplus \underline{0})$) = Length $(x \oplus (y \oplus \underline{0})) \lor \neg$ (Lengthx) + (Length $(y \oplus \underline{1})$) = Length $(x \oplus (y \oplus \underline{1}))$ from H:x:y

- 2: Length $(y \oplus \underline{0}) = S(\text{Length}y)$ from <u>259</u>;y
- 3: Length $(y \oplus \underline{1}) = S(\text{Length}y)$ from <u>259</u>; y
- 4: $x \oplus (y \oplus \underline{0}) = (x \oplus y) \oplus \underline{0}$ from <u>183</u>;x;y;<u>0</u>
- 5: $x \oplus (y \oplus \underline{1}) = (x \oplus y) \oplus \underline{1}$ from <u>183</u>;x;y;<u>1</u>
- 6: Length $((x \oplus y) \oplus \underline{0}) = S(Length(x \oplus y))$ from <u>259</u>; $x \oplus y$
- 7: Length $((x \oplus y) \oplus \underline{1}) = S(Length(x \oplus y))$ from <u>259</u>; $x \oplus y$
- 8: S((Lengthx)+(Lengthy)) = (Lengthx)+(S(Lengthy)) from <u>12</u>;Lengthx;Lengthy

Equality substitutions:

9: \neg (Lengthx) + (Lengthy) = Length($x \oplus y$) $\lor \neg$ S((Lengthx) + (Lengthy)) = (Lengthx) + (S(Lengthy)) \lor S(Length($x \oplus y$)) = (Lengthx) + (S(Lengthy))

10: $\neg \text{Length}(y \oplus \underline{0}) = S(\text{Length}y) \lor (\text{Length}x) + (\frac{\text{Length}(y \oplus \underline{0})}{(\text{Length}x) + (S(\text{Length}y))} = S(\text{Length}(x \oplus y))$

11: $\neg \text{Length}(y \oplus \underline{1}) = \text{S}(\text{Length}y) \lor (\text{Length}x) + (\frac{\text{Length}(y \oplus \underline{1})}{\text{Length}(x \oplus y)}) = \text{S}(\text{Length}(x \oplus y))$ $\lor \neg (\text{Length}x) + (\frac{\text{S}(\text{Length}y)}{\text{S}(\text{Length}(x \oplus y))})$

12: $\neg x \oplus (y \oplus \underline{0}) = (x \oplus y) \oplus \underline{0} \lor \operatorname{Length}(x \oplus (y \oplus \underline{0})) = \operatorname{S}(\operatorname{Length}(x \oplus y))$ $\lor \neg \operatorname{Length}((x \oplus y) \oplus \underline{0}) = \operatorname{S}(\operatorname{Length}(x \oplus y))$

13: $\neg x \oplus (y \oplus \underline{1}) = (x \oplus y) \oplus \underline{1} \lor \operatorname{Length}(x \oplus (y \oplus \underline{1})) = \operatorname{S}(\operatorname{Length}(x \oplus y))$ $\lor \neg \operatorname{Length}((x \oplus y) \oplus \underline{1}) = \operatorname{S}(\operatorname{Length}(x \oplus y))$ 14: $\neg (\text{Length}(x) + (\text{Length}(y \oplus \underline{0}))) = S(\text{Length}(x \oplus y)) \lor (\text{Length}(x) + (\text{Length}(y \oplus \underline{0}))) = \text{Length}(x \oplus (y \oplus \underline{0})) \lor \neg S(\text{Length}(x \oplus y)) = \text{Length}(x \oplus (y \oplus \underline{0}))$

15: $\neg (\text{Length}(x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y)) \lor (\text{Length}(x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1})) \lor \neg S(\text{Length}(x \oplus y)) = \text{Length}(x \oplus (y \oplus \underline{1}))$

Inferences:

16: \neg S((Lengthx) + (Lengthy)) = (Lengthx) + (S(Lengthy)) \lor (Lengthx) + $(S(Lengthy)) = S(Length(x \oplus y))$ bv 0: $(\text{Length}x) + (\text{Length}y) = \text{Length}(x \oplus y)$ 9: \neg (Lengthx) + (Lengthy) = Length(x \oplus y) \lor \neg S((Lengthx) + (Lengthy)) = $(\text{Length}x) + (\text{S}(\text{Length}y)) \lor (\text{Length}x) + (\text{S}(\text{Length}y)) = \text{S}(\text{Length}(x \oplus y))$ 17: $(\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = S(\text{Length}(x \oplus y)) \lor \neg (\text{Length}x) + (S(\text{Length}y)) =$ $S(Length(x \oplus y))$ by 2: Length $(y \oplus 0) = S(\text{Length}y)$ 10: $\neg \text{Length}(y \oplus \underline{0}) = S(\text{Length}y) \lor (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = S(\text{Length}(x \oplus y))$ \lor \neg (Lengthx) + (S(Lengthy)) = S(Length(x \oplus y)) 18: $(\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y)) \lor \neg (\text{Length}x) + (S(\text{Length}y)) =$ $S(Length(x \oplus y))$ by 3: Length $(y \oplus \underline{1}) = S(\text{Length}y)$ 11: $\neg \text{Length}(y \oplus \underline{1}) = S(\text{Length}y) \lor (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y))$ \lor \neg (Lengthx) + (S(Lengthy)) = S(Length(x \oplus y)) 19: Length $(x \oplus (y \oplus \underline{0})) = S(Length(x \oplus y)) \lor \neg Length((x \oplus y) \oplus \underline{0}) = S(Length(x \oplus y))$ by 4: $x \oplus (y \oplus 0) = (x \oplus y) \oplus 0$ 12: $\neg x \oplus (y \oplus \underline{0}) = (x \oplus y) \oplus \underline{0} \quad \lor \quad \text{Length}(x \oplus (y \oplus \underline{0})) = S(\text{Length}(x \oplus y))$ $\lor \neg \text{Length}((x \oplus y) \oplus 0) = S(\text{Length}(x \oplus y))$ 20: Length $(x \oplus (y \oplus 1)) = S(Length(x \oplus y)) \lor \neg Length((x \oplus y) \oplus 1) = S(Length(x \oplus y))$ by 5: $x \oplus (y \oplus \underline{1}) = (x \oplus y) \oplus \underline{1}$ 13: $\neg x \oplus (y \oplus \underline{1}) = (x \oplus y) \oplus \underline{1} \quad \lor \quad \text{Length}(x \oplus (y \oplus \underline{1})) = \text{S}(\text{Length}(x \oplus y))$ $\lor \neg \text{Length}((x \oplus y) \oplus 1) = S(\text{Length}(x \oplus y))$ 21: Length $(x \oplus (y \oplus 0)) = S(\text{Length}(x \oplus y))$ by 6: Length $((x \oplus y) \oplus \underline{0}) = S(Length(x \oplus y))$

19: Length $(x \oplus (y \oplus 0)) = S(\text{Length}(x \oplus y)) \lor \neg \text{Length}((x \oplus y) \oplus 0) =$ $S(Length(x \oplus y))$ 22: Length $(x \oplus (y \oplus \underline{1})) = S(\text{Length}(x \oplus y))$ by 7: Length $((x \oplus y) \oplus \underline{1}) = S(\text{Length}(x \oplus y))$ 20: Length $(x \oplus (y \oplus \underline{1})) = S(Length(x \oplus y)) \lor \neg Length((x \oplus y) \oplus \underline{1}) =$ $S(Length(x \oplus y))$ 23: $(\text{Length}x) + (S(\text{Length}y)) = S(\text{Length}(x \oplus y))$ by 8: S((Lengthx) + (Lengthy)) = (Lengthx) + (S(Lengthy))16: \neg S((Lengthx) + (Lengthy)) = (Lengthx) + (S(Lengthy)) \lor (Lengthx) + $(S(Lengthy)) = S(Length(x \oplus y))$ 24: \neg (Lengthx) + (Length $(y \oplus \underline{0})$) = S(Length $(x \oplus y)$) \lor (Lengthx) + (Length $(y \oplus \underline{0})$) = Length($x \oplus (y \oplus 0)$) by 21: Length $(x \oplus (y \oplus \underline{0})) = S(Length(x \oplus y))$ 14: \neg (Lengthx) + (Length $(y \oplus \underline{0})$) = S(Length $(x \oplus y)$) \lor (Lengthx) + (Length $(y \oplus \underline{0})$) $0)) = \text{Length}(x \oplus (y \oplus 0)) \quad \lor \quad \neg \text{Length}(x \oplus (y \oplus 0)) = \text{S}(\text{Length}(x \oplus y))$ 25: $\neg (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y)) \lor (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) =$ Length $(x \oplus (y \oplus \underline{1}))$ by 22: Length $(x \oplus (y \oplus 1)) = S(Length(x \oplus y))$ 15: \neg (Lengthx) + (Length $(y \oplus 1)$) = S(Length $(x \oplus y)$) \lor (Lengthx) + (Length $(y \oplus 1)$) $\underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1})) \quad \lor \quad \neg \text{Length}(x \oplus (y \oplus \underline{1})) = \text{S}(\text{Length}(x \oplus y))$ 26: $(\text{Length}x) + (\text{Length}(y \oplus 0)) = S(\text{Length}(x \oplus y))$ by 23: $(\text{Length}x) + (S(\text{Length}y)) = S(\text{Length}(x \oplus y))$ 17: $(\text{Length}x) + (\text{Length}(y \oplus 0)) = S(\text{Length}(x \oplus y)) \lor \neg (\text{Length}x) + (S(\text{Length}y)) =$ $S(Length(x \oplus y))$ 27: $(\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y))$ by 23: $(\text{Length}x) + (S(\text{Length}y)) = S(\text{Length}(x \oplus y))$ 18: $(\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y)) \lor \neg (\text{Length}x) + (S(\text{Length}y)) =$ $S(Length(x \oplus y))$ 28: $(\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = \text{Length}(x \oplus (y \oplus \underline{0}))$ by 26: $(\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = S(\text{Length}(x \oplus y))$ 24: \neg (Lengthx) + (Length $(y \oplus \underline{0})$) = S(Length $(x \oplus y)$) \vee (Lengthx) + (Length(y \oplus $(\underline{0})) = \text{Length}(x \oplus (y \oplus \underline{0}))$ 29: $(\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1}))$ by

3

27: $(\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y))$

25: $\neg (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y)) \lor (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) \lor (\text{Length}x)$

- $\underline{1})) = \operatorname{Length}(x \oplus (y \oplus \underline{1}))$
- $\begin{array}{ll} 30: & \neg \left(\operatorname{Length} x\right) + \left(\operatorname{Length} (y \oplus \underline{1})\right) = \operatorname{Length} (x \oplus (y \oplus \underline{1})) & \text{by} \\ & 28: \left(\operatorname{Length} x\right) + \left(\operatorname{Length} (y \oplus \underline{0})\right) = \operatorname{Length} (x \oplus (y \oplus \underline{0})) \\ & 1: \neg \left(\operatorname{Length} x\right) + \left(\operatorname{Length} (y \oplus \underline{0})\right) = \operatorname{Length} (x \oplus (y \oplus \underline{0})) & \lor & \neg \left(\operatorname{Length} x\right) + \left(\operatorname{Length} (y \oplus \underline{1})\right) = \operatorname{Length} (x \oplus (y \oplus \underline{1})) \end{array}$
- 31: QEAby
 - 29: $(\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1}))$
 - 30: \neg (Lengthx) + (Length $(y \oplus \underline{1})$) = Length $(x \oplus (y \oplus \underline{1}))$