

Proof of Theorem 260ij

The theorem to be proved is

$$\text{Length}(x \oplus y) = \text{Length } x + \text{Length } y \rightarrow \text{Length}(x \oplus (y \oplus \underline{0})) = \text{Length } x + \text{Length}(y \oplus \underline{0})$$

$$\& \text{Length}(x \oplus (y \oplus \underline{1})) = \text{Length } x + \text{Length}(y \oplus \underline{1})$$

Suppose the theorem does not hold. Then, with the variables held fixed,

$$(H) \quad [(\text{Length}(x \oplus y)) = ((\text{Length}x) + (\text{Length}y))] \quad \& \quad [\neg (\text{Length}(x \oplus (y \oplus \underline{0}))) = ((\text{Length}x) + (\text{Length}(y \oplus \underline{0}))) \vee \neg (\text{Length}(x \oplus (y \oplus \underline{1}))) = ((\text{Length}x) + (\text{Length}(y \oplus \underline{1})))]$$

Special cases of the hypothesis and previous results:

- 0: $(\text{Length}x) + (\text{Length}y) = \text{Length}(x \oplus y)$ from H: $x:y$
- 1: $\neg (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = \text{Length}(x \oplus (y \oplus \underline{0})) \vee \neg (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1}))$ from H: $x:y$
- 2: $\text{Length}(y \oplus \underline{0}) = S(\text{Length}y)$ from [259](#); y
- 3: $\text{Length}(y \oplus \underline{1}) = S(\text{Length}y)$ from [259](#); y
- 4: $x \oplus (y \oplus \underline{0}) = (x \oplus y) \oplus \underline{0}$ from [183](#); $x;y;\underline{0}$
- 5: $x \oplus (y \oplus \underline{1}) = (x \oplus y) \oplus \underline{1}$ from [183](#); $x;y;\underline{1}$
- 6: $\text{Length}((x \oplus y) \oplus \underline{0}) = S(\text{Length}(x \oplus y))$ from [259](#); $x \oplus y$
- 7: $\text{Length}((x \oplus y) \oplus \underline{1}) = S(\text{Length}(x \oplus y))$ from [259](#); $x \oplus y$
- 8: $S((\text{Length}x) + (\text{Length}y)) = (\text{Length}x) + (S(\text{Length}y))$ from [12](#); $\text{Length}x;\text{Length}y$

Equality substitutions:

- 9: $\neg (\text{Length}x) + (\text{Length}y) = \text{Length}(x \oplus y) \vee \neg S((\text{Length}x) + (\text{Length}y)) = (\text{Length}x) + (S(\text{Length}y)) \vee S(\text{Length}(x \oplus y)) = (\text{Length}x) + (S(\text{Length}y))$
- 10: $\neg \text{Length}(y \oplus \underline{0}) = S(\text{Length}y) \vee (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = S(\text{Length}(x \oplus y)) \vee \neg (\text{Length}x) + (S(\text{Length}y)) = S(\text{Length}(x \oplus y))$
- 11: $\neg \text{Length}(y \oplus \underline{1}) = S(\text{Length}y) \vee (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y)) \vee \neg (\text{Length}x) + (S(\text{Length}y)) = S(\text{Length}(x \oplus y))$
- 12: $\neg x \oplus (y \oplus \underline{0}) = (x \oplus y) \oplus \underline{0} \vee \text{Length}(x \oplus (y \oplus \underline{0})) = S(\text{Length}(x \oplus y)) \vee \neg \text{Length}((x \oplus y) \oplus \underline{0}) = S(\text{Length}(x \oplus y))$
- 13: $\neg x \oplus (y \oplus \underline{1}) = (x \oplus y) \oplus \underline{1} \vee \text{Length}(x \oplus (y \oplus \underline{1})) = S(\text{Length}(x \oplus y)) \vee \neg \text{Length}((x \oplus y) \oplus \underline{1}) = S(\text{Length}(x \oplus y))$

$$14: \neg (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = \text{S}(\text{Length}(x \oplus y)) \vee (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = \text{Length}(x \oplus (y \oplus \underline{0})) \vee \neg \text{S}(\text{Length}(x \oplus y)) = \text{Length}(x \oplus (y \oplus \underline{0}))$$

$$15: \neg (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{S}(\text{Length}(x \oplus y)) \vee (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1})) \vee \neg \text{S}(\text{Length}(x \oplus y)) = \text{Length}(x \oplus (y \oplus \underline{1}))$$

Inferences:

$$16: \neg \text{S}((\text{Length}x) + (\text{Length}y)) = (\text{Length}x) + (\text{S}(\text{Length}y)) \vee (\text{Length}x) + (\text{S}(\text{Length}y)) = \text{S}(\text{Length}(x \oplus y)) \quad \text{by}$$

$$0: (\text{Length}x) + (\text{Length}y) = \text{Length}(x \oplus y)$$

$$9: \neg (\text{Length}x) + (\text{Length}y) = \text{Length}(x \oplus y) \vee \neg \text{S}((\text{Length}x) + (\text{Length}y)) = (\text{Length}x) + (\text{S}(\text{Length}y)) \vee (\text{Length}x) + (\text{S}(\text{Length}y)) = \text{S}(\text{Length}(x \oplus y))$$

$$17: (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = \text{S}(\text{Length}(x \oplus y)) \vee \neg (\text{Length}x) + (\text{S}(\text{Length}y)) = \text{S}(\text{Length}(x \oplus y)) \quad \text{by}$$

$$2: \text{Length}(y \oplus \underline{0}) = \text{S}(\text{Length}y)$$

$$10: \neg \text{Length}(y \oplus \underline{0}) = \text{S}(\text{Length}y) \vee (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = \text{S}(\text{Length}(x \oplus y)) \vee \neg (\text{Length}x) + (\text{S}(\text{Length}y)) = \text{S}(\text{Length}(x \oplus y))$$

$$18: (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{S}(\text{Length}(x \oplus y)) \vee \neg (\text{Length}x) + (\text{S}(\text{Length}y)) = \text{S}(\text{Length}(x \oplus y)) \quad \text{by}$$

$$3: \text{Length}(y \oplus \underline{1}) = \text{S}(\text{Length}y)$$

$$11: \neg \text{Length}(y \oplus \underline{1}) = \text{S}(\text{Length}y) \vee (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{S}(\text{Length}(x \oplus y)) \vee \neg (\text{Length}x) + (\text{S}(\text{Length}y)) = \text{S}(\text{Length}(x \oplus y))$$

$$19: \text{Length}(x \oplus (y \oplus \underline{0})) = \text{S}(\text{Length}(x \oplus y)) \vee \neg \text{Length}((x \oplus y) \oplus \underline{0}) = \text{S}(\text{Length}(x \oplus y)) \quad \text{by}$$

$$4: x \oplus (y \oplus \underline{0}) = (x \oplus y) \oplus \underline{0}$$

$$12: \neg x \oplus (y \oplus \underline{0}) = (x \oplus y) \oplus \underline{0} \vee \text{Length}(x \oplus (y \oplus \underline{0})) = \text{S}(\text{Length}(x \oplus y)) \vee \neg \text{Length}((x \oplus y) \oplus \underline{0}) = \text{S}(\text{Length}(x \oplus y))$$

$$20: \text{Length}(x \oplus (y \oplus \underline{1})) = \text{S}(\text{Length}(x \oplus y)) \vee \neg \text{Length}((x \oplus y) \oplus \underline{1}) = \text{S}(\text{Length}(x \oplus y)) \quad \text{by}$$

$$5: x \oplus (y \oplus \underline{1}) = (x \oplus y) \oplus \underline{1}$$

$$13: \neg x \oplus (y \oplus \underline{1}) = (x \oplus y) \oplus \underline{1} \vee \text{Length}(x \oplus (y \oplus \underline{1})) = \text{S}(\text{Length}(x \oplus y)) \vee \neg \text{Length}((x \oplus y) \oplus \underline{1}) = \text{S}(\text{Length}(x \oplus y))$$

$$21: \text{Length}(x \oplus (y \oplus \underline{0})) = \text{S}(\text{Length}(x \oplus y)) \quad \text{by}$$

$$6: \text{Length}((x \oplus y) \oplus \underline{0}) = \text{S}(\text{Length}(x \oplus y))$$

$$27: (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y))$$

$$25: \neg (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = S(\text{Length}(x \oplus y)) \quad \vee \quad (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1}))$$

$$30: \neg (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1})) \quad \text{by}$$

$$28: (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = \text{Length}(x \oplus (y \oplus \underline{0}))$$

$$1: \neg (\text{Length}x) + (\text{Length}(y \oplus \underline{0})) = \text{Length}(x \oplus (y \oplus \underline{0})) \quad \vee \quad \neg (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1}))$$

$$31: \text{QEA} \quad \text{by}$$

$$29: (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1}))$$

$$30: \neg (\text{Length}x) + (\text{Length}(y \oplus \underline{1})) = \text{Length}(x \oplus (y \oplus \underline{1}))$$