Proof of Theorem 25

The theorem to be proved is

$x \neq 0 \quad \& \quad \mathrm{P}x = 0 \quad \rightarrow \quad x = \mathrm{S}0$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\,(x) = (0)] \quad \& \quad [(\mathrm{P}x) = (0)] \quad \& \quad [\neg\,(x) = (\mathrm{S}0)]]$

Special cases of the hypothesis and previous results:

0: $\neg\,0 = x$ from H:x

1: $\mathrm{P}x = 0$ from H:x

2: $\neg\,\mathrm{S}0 = x$ from H:x

3: $0 = x \quad \vee \quad \mathrm{S}(\mathrm{P}x) = x$ from 22;x

Equality substitutions:

4: $\neg\,\mathrm{P}x = 0 \quad \vee \quad \neg\,\mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \mathrm{S}(0) = x$

Inferences:

5: $\mathrm{S}(\mathrm{P}x) = x$ by
0: $\neg\,0 = x$
3: $0 = x \quad \vee \quad \mathrm{S}(\mathrm{P}x) = x$

6: $\neg\,\mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \mathrm{S}0 = x$ by
1: $\mathrm{P}x = 0$
4: $\neg\,\mathrm{P}x = 0 \quad \vee \quad \neg\,\mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \mathrm{S}0 = x$

7: $\neg\,\mathrm{S}(\mathrm{P}x) = x$ by
2: $\neg\,\mathrm{S}0 = x$
6: $\neg\,\mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \mathrm{S}0 = x$

8: QEA by
5: $\mathrm{S}(\mathrm{P}x) = x$
7: $\neg\,\mathrm{S}(\mathrm{P}x) = x$