

Proof of Theorem 241i

The theorem to be proved is

$$\text{Half}(x \cdot 2 + 1) = x \quad \& \quad \text{Half}(x \cdot 2) = x \quad \rightarrow \quad \text{Half}(Sx \cdot 2 + 1) = Sx \quad \& \quad \text{Half}(Sx \cdot 2) = Sx$$

Suppose the theorem does not hold. Then, with the variables held fixed,

$$\begin{aligned} \text{(H)} \quad & [(\text{Half}((x \cdot 2) + 1)) = (x)] \quad \& \quad [(\text{Half}(x \cdot 2)) = (x)] \quad \& \quad [\neg (\text{Half}(((Sx) \cdot 2) + 1)) = (Sx)] \\ & \vee \quad \neg (\text{Half}((Sx) \cdot 2)) = (Sx)] \end{aligned}$$

Special cases of the hypothesis and previous results:

- 0: $\text{Half}((x \cdot 2) + 1) = x$ from H: x
- 1: $\neg \text{Half}(((Sx) \cdot 2) + 1) = Sx \quad \vee \quad \neg \text{Half}((Sx) \cdot 2) = Sx$ from H: x
- 2: $S(S0) = 2$ from [116](#)
- 3: $S0 = 1$ from [115](#)
- 4: $(x \cdot 2) + 2 = (Sx) \cdot 2$ from [104](#); $x;2$
- 5: $S((x \cdot 2) + 1) = (x \cdot 2) + (S1)$ from [12](#); $x \cdot 2;1$
- 6: $\text{Parity}((x \cdot 2) + 1) = \text{Parity}1$ from [212](#); $x;1$
- 7: $\text{Parity}1 = 1$ from [208](#)
- 8: $\text{Parity}2 = 0$ from [208](#)
- 9: $\neg \text{Parity}((x \cdot 2) + 1) = 1 \quad \vee \quad \text{Half}(S((x \cdot 2) + 1)) = S(\text{Half}((x \cdot 2) + 1))$ from [220](#); $(x \cdot 2) + 1$
- 10: $((x \cdot 2) + 2) + 0 = (x \cdot 2) + 2$ from [12](#); $(x \cdot 2) + 2;0$
- 11: $S(((x \cdot 2) + 2) + 0) = ((x \cdot 2) + 2) + (S0)$ from [12](#); $(x \cdot 2) + 2;0$
- 12: $\text{Parity}((x \cdot 2) + 2) = \text{Parity}2$ from [212](#); $x;2$
- 13: $\neg \text{Parity}((x \cdot 2) + 2) = 0 \quad \vee \quad \text{Half}(S((x \cdot 2) + 2)) = \text{Half}((x \cdot 2) + 2)$ from [219](#); $(x \cdot 2) + 2$

Equality substitutions:

- 14: $\neg \text{Half}((x \cdot 2) + 1) = x \quad \vee \quad \neg \text{Half}(S((x \cdot 2) + 1)) = S(\text{Half}((x \cdot 2) + 1))$
 $\vee \quad \text{Half}(S((x \cdot 2) + 1)) = S(x)$
- 15: $\neg \text{Half}((Sx) \cdot 2) = Sx \quad \vee \quad \neg \text{Half}(S((Sx) \cdot 2)) = \text{Half}((Sx) \cdot 2) \quad \vee \quad \text{Half}(S((Sx) \cdot 2)) = Sx$
- 16: $\neg S0 = 1 \quad \vee \quad \neg S(S0) = 2 \quad \vee \quad S(1) = 2$

$$17: \neg S0 = 1 \quad \vee \quad \neg S((x \cdot 2) + 2) + 0 = ((x \cdot 2) + 2) + (S0) \quad \vee \quad S((x \cdot 2) + 2) + 0 = ((x \cdot 2) + 2) + (1)$$

$$18: \neg (x \cdot 2) + 2 = (Sx) \cdot 2 \quad \vee \quad \neg ((x \cdot 2) + 2) + 0 = (x \cdot 2) + 2 \quad \vee \quad ((Sx) \cdot 2) + 0 = (Sx) \cdot 2$$

$$19: \neg (x \cdot 2) + 2 = (Sx) \cdot 2 \quad \vee \quad \neg \text{Half}(S((x \cdot 2) + 2)) = \text{Half}((x \cdot 2) + 2) \quad \vee \quad \text{Half}(S((Sx) \cdot 2)) = \text{Half}((Sx) \cdot 2)$$

$$20: \neg (x \cdot 2) + 2 = (Sx) \cdot 2 \quad \vee \quad \neg S(((x \cdot 2) + 2) + 0) = ((x \cdot 2) + 2) + 1 \quad \vee \quad S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1$$

$$21: \neg (x \cdot 2) + 2 = (Sx) \cdot 2 \quad \vee \quad \neg \text{Half}((x \cdot 2) + (S1)) = \text{Half}((x \cdot 2) + 2) \quad \vee \quad \text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$$

$$22: \neg S((x \cdot 2) + 1) = (x \cdot 2) + (S1) \quad \vee \quad \text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2) \quad \vee \quad \neg \text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$$

$$23: \neg \text{Parity}((x \cdot 2) + 1) = \text{Parity1} \quad \vee \quad \text{Parity}((x \cdot 2) + 1) = 1 \quad \vee \quad \neg \text{Parity1} = 1$$

$$24: \neg \text{Parity2} = 0 \quad \vee \quad \neg \text{Parity}((x \cdot 2) + 2) = \text{Parity2} \quad \vee \quad \text{Parity}((x \cdot 2) + 2) = 0$$

$$25: \neg \text{Half}(S((x \cdot 2) + 1)) = Sx \quad \vee \quad \neg \text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2) \quad \vee \quad Sx = \text{Half}((Sx) \cdot 2)$$

$$26: \neg S1 = 2 \quad \vee \quad \text{Half}((x \cdot 2) + (S1)) = \text{Half}((x \cdot 2) + 2) \quad \vee \quad \neg \text{Half}((x \cdot 2) + (2)) = \text{Half}((x \cdot 2) + 2)$$

$$27: \neg ((Sx) \cdot 2) + 0 = (Sx) \cdot 2 \quad \vee \quad \neg S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1 \quad \vee \quad S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1$$

$$28: \neg ((Sx) \cdot 2) + 1 = S((Sx) \cdot 2) \quad \vee \quad \text{Half}(((Sx) \cdot 2) + 1) = Sx \quad \vee \quad \neg \text{Half}(S((Sx) \cdot 2)) = Sx$$

Inferences:

$$29: \neg \text{Half}(S((x \cdot 2) + 1)) = S(\text{Half}((x \cdot 2) + 1)) \quad \vee \quad \text{Half}(S((x \cdot 2) + 1)) = Sx \quad \text{by}$$

$$0: \text{Half}((x \cdot 2) + 1) = x$$

$$14: \neg \text{Half}((x \cdot 2) + 1) = x \quad \vee \quad \neg \text{Half}(S((x \cdot 2) + 1)) = S(\text{Half}((x \cdot 2) + 1))$$

$$\vee \quad \text{Half}(S((x \cdot 2) + 1)) = Sx$$

$$30: \neg S0 = 1 \quad \vee \quad S1 = 2 \quad \text{by}$$

$$2: S(S0) = 2$$

$$16: \neg S0 = 1 \quad \vee \quad \neg S(S0) = 2 \quad \vee \quad S1 = 2$$

31: $\neg S((x \cdot 2) + 2) + 0 = ((x \cdot 2) + 2) + (S0) \quad \vee \quad S((x \cdot 2) + 2) + 0 = ((x \cdot 2) + 2) + 1$
by

3: $S0 = 1$

17: $\neg S0 = 1 \quad \vee \quad \neg S((x \cdot 2) + 2) + 0 = ((x \cdot 2) + 2) + (S0) \quad \vee \quad S((x \cdot 2) + 2) + 0 = ((x \cdot 2) + 2) + 1$

32: $S1 = 2$ by

3: $S0 = 1$

30: $\neg S0 = 1 \quad \vee \quad S1 = 2$

33: $\neg ((x \cdot 2) + 2) + 0 = (x \cdot 2) + 2 \quad \vee \quad ((Sx) \cdot 2) + 0 = (Sx) \cdot 2$ by

4: $(x \cdot 2) + 2 = (Sx) \cdot 2$

18: $\neg (x \cdot 2) + 2 = (Sx) \cdot 2 \quad \vee \quad \neg ((x \cdot 2) + 2) + 0 = (x \cdot 2) + 2 \quad \vee \quad ((Sx) \cdot 2) + 0 = (Sx) \cdot 2$

34: $\neg \text{Half}(S((x \cdot 2) + 2)) = \text{Half}((x \cdot 2) + 2) \quad \vee \quad \text{Half}(S((Sx) \cdot 2)) = \text{Half}((Sx) \cdot 2)$ by

4: $(x \cdot 2) + 2 = (Sx) \cdot 2$

19: $\neg (x \cdot 2) + 2 = (Sx) \cdot 2 \quad \vee \quad \neg \text{Half}(S((x \cdot 2) + 2)) = \text{Half}((x \cdot 2) + 2)$
 $\vee \quad \text{Half}(S((Sx) \cdot 2)) = \text{Half}((Sx) \cdot 2)$

35: $\neg S((x \cdot 2) + 2) + 0 = ((x \cdot 2) + 2) + 1 \quad \vee \quad S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1$ by

4: $(x \cdot 2) + 2 = (Sx) \cdot 2$

20: $\neg (x \cdot 2) + 2 = (Sx) \cdot 2 \quad \vee \quad \neg S((x \cdot 2) + 2) + 0 = ((x \cdot 2) + 2) + 1$
 $\vee \quad S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1$

36: $\neg \text{Half}((x \cdot 2) + (S1)) = \text{Half}((x \cdot 2) + 2) \quad \vee \quad \text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$
by

4: $(x \cdot 2) + 2 = (Sx) \cdot 2$

21: $\neg (x \cdot 2) + 2 = (Sx) \cdot 2 \quad \vee \quad \neg \text{Half}((x \cdot 2) + (S1)) = \text{Half}((x \cdot 2) + 2)$
 $\vee \quad \text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$

37: $\text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2) \quad \vee \quad \neg \text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$ by

5: $S((x \cdot 2) + 1) = (x \cdot 2) + (S1)$

22: $\neg S((x \cdot 2) + 1) = (x \cdot 2) + (S1) \quad \vee \quad \text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2)$
 $\vee \quad \neg \text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$

38: $\text{Parity}((x \cdot 2) + 1) = 1 \quad \vee \quad \neg \text{Parity}1 = 1$ by

6: $\text{Parity}((x \cdot 2) + 1) = \text{Parity}1$

23: $\neg \text{Parity}((x \cdot 2) + 1) = \text{Parity}1 \quad \vee \quad \text{Parity}((x \cdot 2) + 1) = 1 \quad \vee \quad \neg \text{Parity}1 = 1$

39: $\text{Parity}((x \cdot 2) + 1) = 1$ by

7: $\text{Parity}1 = 1$

- 38: $\text{Parity}((x \cdot 2) + 1) = 1 \quad \vee \quad \neg \text{Parity1} = 1$
- 40: $\neg \text{Parity}((x \cdot 2) + 2) = \text{Parity2} \quad \vee \quad \text{Parity}((x \cdot 2) + 2) = 0 \quad \text{by}$
8: $\text{Parity2} = 0$
- 24: $\neg \text{Parity2} = 0 \quad \vee \quad \neg \text{Parity}((x \cdot 2) + 2) = \text{Parity2} \quad \vee \quad \text{Parity}((x \cdot 2) + 2) = 0$
- 41: $((Sx) \cdot 2) + 0 = (Sx) \cdot 2 \quad \text{by}$
10: $((x \cdot 2) + 2) + 0 = (x \cdot 2) + 2$
33: $\neg ((x \cdot 2) + 2) + 0 = (x \cdot 2) + 2 \quad \vee \quad ((Sx) \cdot 2) + 0 = (Sx) \cdot 2$
- 42: $S(((x \cdot 2) + 2) + 0) = ((x \cdot 2) + 2) + 1 \quad \text{by}$
11: $S(((x \cdot 2) + 2) + 0) = ((x \cdot 2) + 2) + (S0)$
31: $\neg S(((x \cdot 2) + 2) + 0) = ((x \cdot 2) + 2) + (S0) \quad \vee \quad S(((x \cdot 2) + 2) + 0) = ((x \cdot 2) + 2) + 1$
- 43: $\text{Parity}((x \cdot 2) + 2) = 0 \quad \text{by}$
12: $\text{Parity}((x \cdot 2) + 2) = \text{Parity2}$
40: $\neg \text{Parity}((x \cdot 2) + 2) = \text{Parity2} \quad \vee \quad \text{Parity}((x \cdot 2) + 2) = 0$
- 44: $\text{Half}((x \cdot 2) + (S1)) = \text{Half}((x \cdot 2) + 2) \quad \text{by}$
32: $S1 = 2$
26: $\neg S1 = 2 \quad \vee \quad \text{Half}((x \cdot 2) + (S1)) = \text{Half}((x \cdot 2) + 2)$
- 45: $\text{Half}(S((x \cdot 2) + 1)) = S(\text{Half}((x \cdot 2) + 1)) \quad \text{by}$
39: $\text{Parity}((x \cdot 2) + 1) = 1$
9: $\neg \text{Parity}((x \cdot 2) + 1) = 1 \quad \vee \quad \text{Half}(S((x \cdot 2) + 1)) = S(\text{Half}((x \cdot 2) + 1))$
- 46: $\neg S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1 \quad \vee \quad ((Sx) \cdot 2) + 1 = S((Sx) \cdot 2) \quad \text{by}$
41: $((Sx) \cdot 2) + 0 = (Sx) \cdot 2$
27: $\neg ((Sx) \cdot 2) + 0 = (Sx) \cdot 2 \quad \vee \quad \neg S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1 \quad \vee \quad ((Sx) \cdot 2) + 1 = S((Sx) \cdot 2)$
- 47: $S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1 \quad \text{by}$
42: $S(((x \cdot 2) + 2) + 0) = ((x \cdot 2) + 2) + 1$
35: $\neg S(((x \cdot 2) + 2) + 0) = ((x \cdot 2) + 2) + 1 \quad \vee \quad S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1$
- 48: $\text{Half}(S((x \cdot 2) + 2)) = \text{Half}((x \cdot 2) + 2) \quad \text{by}$
43: $\text{Parity}((x \cdot 2) + 2) = 0$
13: $\neg \text{Parity}((x \cdot 2) + 2) = 0 \quad \vee \quad \text{Half}(S((x \cdot 2) + 2)) = \text{Half}((x \cdot 2) + 2)$
- 49: $\text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2) \quad \text{by}$
44: $\text{Half}((x \cdot 2) + (S1)) = \text{Half}((x \cdot 2) + 2)$
36: $\neg \text{Half}((x \cdot 2) + (S1)) = \text{Half}((x \cdot 2) + 2) \quad \vee \quad \text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$

50: $\text{Half}(S((x \cdot 2) + 1)) = Sx$ by
 45: $\text{Half}(S((x \cdot 2) + 1)) = S(\text{Half}((x \cdot 2) + 1))$
 29: $\neg \text{Half}(S((x \cdot 2) + 1)) = S(\text{Half}((x \cdot 2) + 1)) \vee \text{Half}(S((x \cdot 2) + 1)) = Sx$

51: $((Sx) \cdot 2) + 1 = S((Sx) \cdot 2)$ by
 47: $S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1$
 46: $\neg S(((Sx) \cdot 2) + 0) = ((Sx) \cdot 2) + 1 \vee ((Sx) \cdot 2) + 1 = S((Sx) \cdot 2)$

52: $\text{Half}(S((Sx) \cdot 2)) = \text{Half}((Sx) \cdot 2)$ by
 48: $\text{Half}(S((x \cdot 2) + 2)) = \text{Half}((x \cdot 2) + 2)$
 34: $\neg \text{Half}(S((x \cdot 2) + 2)) = \text{Half}((x \cdot 2) + 2) \vee \text{Half}(S((Sx) \cdot 2)) = \text{Half}((Sx) \cdot 2)$

53: $\text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2)$ by
 49: $\text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$
 37: $\text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2) \vee \neg \text{Half}((x \cdot 2) + (S1)) = \text{Half}((Sx) \cdot 2)$

54: $\neg \text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2) \vee \text{Half}((Sx) \cdot 2) = Sx$ by
 50: $\text{Half}(S((x \cdot 2) + 1)) = Sx$
 25: $\neg \text{Half}(S((x \cdot 2) + 1)) = Sx \vee \neg \text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2)$
 $\vee \text{Half}((Sx) \cdot 2) = Sx$

55: $\text{Half}(((Sx) \cdot 2) + 1) = Sx \vee \neg \text{Half}(S((Sx) \cdot 2)) = Sx$ by
 51: $((Sx) \cdot 2) + 1 = S((Sx) \cdot 2)$
 28: $\neg ((Sx) \cdot 2) + 1 = S((Sx) \cdot 2) \vee \text{Half}(((Sx) \cdot 2) + 1) = Sx \vee \neg \text{Half}(S((Sx) \cdot 2)) = Sx$

56: $\neg \text{Half}((Sx) \cdot 2) = Sx \vee \text{Half}(S((Sx) \cdot 2)) = Sx$ by
 52: $\text{Half}(S((Sx) \cdot 2)) = \text{Half}((Sx) \cdot 2)$
 15: $\neg \text{Half}((Sx) \cdot 2) = Sx \vee \neg \text{Half}(S((Sx) \cdot 2)) = \text{Half}((Sx) \cdot 2) \vee \text{Half}(S((Sx) \cdot 2)) = Sx$

57: $\text{Half}((Sx) \cdot 2) = Sx$ by
 53: $\text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2)$
 54: $\neg \text{Half}(S((x \cdot 2) + 1)) = \text{Half}((Sx) \cdot 2) \vee \text{Half}((Sx) \cdot 2) = Sx$

58: $\neg \text{Half}(((Sx) \cdot 2) + 1) = Sx$ by
 57: $\text{Half}((Sx) \cdot 2) = Sx$
 1: $\neg \text{Half}(((Sx) \cdot 2) + 1) = Sx \vee \neg \text{Half}((Sx) \cdot 2) = Sx$

59: $\text{Half}(S((Sx) \cdot 2)) = Sx$ by
 57: $\text{Half}((Sx) \cdot 2) = Sx$
 56: $\neg \text{Half}((Sx) \cdot 2) = Sx \vee \text{Half}(S((Sx) \cdot 2)) = Sx$

- 60: $\neg \text{Half}(S((Sx) \cdot 2)) = Sx$ by
- 58: $\neg \text{Half}(((Sx) \cdot 2) + 1) = Sx$
- 55: $\text{Half}(((Sx) \cdot 2) + 1) = Sx \vee \neg \text{Half}(S((Sx) \cdot 2)) = Sx$
- 61: *QEA* by
- 59: $\text{Half}(S((Sx) \cdot 2)) = Sx$
- 60: $\neg \text{Half}(S((Sx) \cdot 2)) = Sx$