Proof of Theorem 23i

The theorem to be proved is

$$[y - x \neq 0 \quad \rightarrow \quad x + (y - x) = y] \quad \rightarrow \quad [y - \mathrm{S}x \neq 0 \quad \rightarrow \quad \mathrm{S}x + (y - \mathrm{S}x) = y]$$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(y - x) = (0) \quad \vee \quad (x + (y - x)) = (y)] \quad \& \quad [\neg\ (y - (\mathrm{S}x)) = (0)] \quad \& \quad [\neg\ ((\mathrm{S}x) + (y - (\mathrm{S}x))) = (y)]]$

Special cases of the hypothesis and previous results:

0: $y - x = 0 \quad \vee \quad x + (y - x) = y$ from H:y:x

1: $\neg\ y - (\mathrm{S}x) = 0$ from H:y:x

2: $\neg\ (\mathrm{S}x) + (y - (\mathrm{S}x)) = y$ from H:y:x

3: $\mathrm{P}(y - x) = y - (\mathrm{S}x)$ from 17;y;x

4: $\mathrm{P}0 = 0$ from 16;$y - x$

5: $(\mathrm{S}x) + (\mathrm{P}(y - x)) = x + (\mathrm{S}(\mathrm{P}(y - x)))$ from 14;x;$\mathrm{P}(y - x)$

6: $y - x = 0 \quad \vee \quad \mathrm{S}(\mathrm{P}(y - x)) = y - x$ from 22;$y - x$

Equality substitutions:

7: $\neg\ y - x = 0 \quad \vee \quad \mathrm{P}(y - x) = 0 \quad \vee \quad \neg\ \mathrm{P}(0) = 0$

8: $\neg\ \mathrm{P}(y - x) = y - (\mathrm{S}x) \quad \vee \quad \neg\ \mathrm{P}(y - x) = 0 \quad \vee \quad y - (\mathrm{S}x) = 0$

9: $\neg\ \mathrm{P}(y - x) = y - (\mathrm{S}x) \quad \vee \quad \neg\ (\mathrm{S}x) + (\mathrm{P}(y - x)) = y \quad \vee \quad (\mathrm{S}x) + (y - (\mathrm{S}x)) = y$

10: $\neg\ (\mathrm{S}x) + (\mathrm{P}(y - x)) = x + (\mathrm{S}(\mathrm{P}(y - x))) \quad \vee \quad (\mathrm{S}x) + (\mathrm{P}(y - x)) = y \quad \vee \quad \neg\ x + (\mathrm{S}(\mathrm{P}(y - x))) = y$

11: $\neg\ \mathrm{S}(\mathrm{P}(y - x)) = y - x \quad \vee \quad x + (\mathrm{S}(\mathrm{P}(y - x))) = y \quad \vee \quad \neg\ x + (y - x) = y$

Inferences:

12: $\neg\ \mathrm{P}(y - x) = y - (\mathrm{S}x) \quad \vee \quad \neg\ \mathrm{P}(y - x) = 0$ by

1: $\neg\ y - (\mathrm{S}x) = 0$

8: $\neg\ \mathrm{P}(y - x) = y - (\mathrm{S}x) \quad \vee \quad \neg\ \mathrm{P}(y - x) = 0 \quad \vee \quad y - (\mathrm{S}x) = 0$

13: $\neg\, \mathrm{P}(y-x) = y-(\mathrm{S}x) \quad \vee \quad \neg\, (\mathrm{S}x)+(\mathrm{P}(y-x)) = y$ by

2: $\neg\, (\mathrm{S}x)+(y-(\mathrm{S}x)) = y$

9: $\neg\, \mathrm{P}(y-x) = y-(\mathrm{S}x) \quad \vee \quad \neg\, (\mathrm{S}x)+(\mathrm{P}(y-x)) = y \quad \vee \quad (\mathrm{S}x)+(y-(\mathrm{S}x)) = y$

14: $\neg\, \mathrm{P}(y-x) = 0$ by

3: $\mathrm{P}(y-x) = y-(\mathrm{S}x)$

12: $\neg\, \mathrm{P}(y-x) = y-(\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{P}(y-x) = 0$

15: $\neg\, (\mathrm{S}x)+(\mathrm{P}(y-x)) = y$ by

3: $\mathrm{P}(y-x) = y-(\mathrm{S}x)$

13: $\neg\, \mathrm{P}(y-x) = y-(\mathrm{S}x) \quad \vee \quad \neg\, (\mathrm{S}x)+(\mathrm{P}(y-x)) = y$

16: $\neg\, y-x = 0 \quad \vee \quad \mathrm{P}(y-x) = 0$ by

4: $\mathrm{P}0 = 0$

7: $\neg\, y-x = 0 \quad \vee \quad \mathrm{P}(y-x) = 0 \quad \vee \quad \neg\, \mathrm{P}0 = 0$

17: $(\mathrm{S}x)+(\mathrm{P}(y-x)) = y \quad \vee \quad \neg\, x+(\mathrm{S}(\mathrm{P}(y-x))) = y$ by

5: $(\mathrm{S}x)+(\mathrm{P}(y-x)) = x+(\mathrm{S}(\mathrm{P}(y-x)))$

10: $\neg\, (\mathrm{S}x)+(\mathrm{P}(y-x)) = x+(\mathrm{S}(\mathrm{P}(y-x))) \quad \vee \quad (\mathrm{S}x)+(\mathrm{P}(y-x)) = y \quad \vee \quad \neg\, x+(\mathrm{S}(\mathrm{P}(y-x))) = y$

18: $\neg\, y-x = 0$ by

14: $\neg\, \mathrm{P}(y-x) = 0$

16: $\neg\, y-x = 0 \quad \vee \quad \mathrm{P}(y-x) = 0$

19: $\neg\, x+(\mathrm{S}(\mathrm{P}(y-x))) = y$ by

15: $\neg\, (\mathrm{S}x)+(\mathrm{P}(y-x)) = y$

17: $(\mathrm{S}x)+(\mathrm{P}(y-x)) = y \quad \vee \quad \neg\, x+(\mathrm{S}(\mathrm{P}(y-x))) = y$

20: $x+(y-x) = y$ by

18: $\neg\, y-x = 0$

0: $y-x = 0 \quad \vee \quad x+(y-x) = y$

21: $\mathrm{S}(\mathrm{P}(y-x)) = y-x$ by

18: $\neg\, y-x = 0$

6: $y-x = 0 \quad \vee \quad \mathrm{S}(\mathrm{P}(y-x)) = y-x$

22: $\neg\, \mathrm{S}(\mathrm{P}(y-x)) = y-x \quad \vee \quad \neg\, x+(y-x) = y$ by

19: $\neg\, x+(\mathrm{S}(\mathrm{P}(y-x))) = y$

11: $\neg\, \mathrm{S}(\mathrm{P}(y-x)) = y-x \quad \vee \quad x+(\mathrm{S}(\mathrm{P}(y-x))) = y \quad \vee \quad \neg\, x+(y-x) = y$

23: $\neg\, \mathrm{S}(\mathrm{P}(y-x)) = y-x$ by

20: $x + (y - x) = y$

22: $\neg\ \mathrm{S}(\mathrm{P}(y - x)) = y - x \quad \vee \quad \neg\ x + (y - x) = y$

24: QEA by

21: $\mathrm{S}(\mathrm{P}(y - x)) = y - x$

23: $\neg\ \mathrm{S}(\mathrm{P}(y - x)) = y - x$