## Proof of Theorem 239

The theorem to be proved is

$x \neq \epsilon \quad \rightarrow \quad \mathrm{Q\,Chop}\,x = \mathrm{Half\,Q}x \quad \& \quad \mathrm{R\,Chop}\,x = \mathrm{Half\,R}x$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\,(x) = (\epsilon)] \quad \& \quad [\neg\,(\mathrm{Q}(\mathrm{Chop}x)) = (\mathrm{Half}(\mathrm{Q}x)) \quad \vee \quad \neg\,(\mathrm{R}(\mathrm{Chop}x)) = (\mathrm{Half}(\mathrm{R}x))]]$

### Special cases of the hypothesis and previous results:

0: $\neg\,\epsilon = x$ from H:$x$

1: $\neg\,\mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \neg\,\mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$ from H:$x$

2: $\neg\,(\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x) \quad \vee \quad \neg\,\mathrm{Half}(\mathrm{Q}x)$ is a power of two $\vee$ $\neg\,\mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x)$ from 171;Chop$x$;Half(Q$x$);Half(R$x$)

3: $\neg\,(\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x) \quad \vee \quad \neg\,\mathrm{Half}(\mathrm{Q}x)$ is a power of two $\vee$ $\neg\,\mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$ from 171;Chop$x$;Half(Q$x$);Half(R$x$)

4: $\epsilon = x \quad \vee \quad (\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x)$ from 231;$x$

5: $\epsilon = x \quad \vee \quad \mathrm{Half}(\mathrm{Q}x)$ is a power of two from 238;$x$

6: $\neg\,\mathrm{Q}x$ is a power of two $\vee$ $\mathrm{Q}x = 1 \quad \vee \quad \mathrm{Parity}(\mathrm{Q}x) = 0$ from 236;Q$x$

7: $\mathrm{Q}x$ is a power of two from 166;$x$

8: $\mathrm{R}x < \mathrm{Q}x$ from 166;$x$

9: $\neg\,\mathrm{Q}x = 1 \quad \vee \quad \epsilon = x$ from 203;$x$

10: $\neg\,\mathrm{Parity}(\mathrm{Q}x) = 0 \quad \vee \quad \neg\,\mathrm{R}x < \mathrm{Q}x \quad \vee \quad \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x)$ from 233;Q$x$;R$x$

### Inferences:

11: $(\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x)$ by

0: $\neg\,\epsilon = x$

4: $\epsilon = x \quad \vee \quad (\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x)$

12: $\mathrm{Half}(\mathrm{Q}x)$ is a power of two by

0: $\neg\,\epsilon = x$

5: $\epsilon = x \quad \vee \quad \mathrm{Half}(\mathrm{Q}x)$ is a power of two

13: $\neg\,\mathrm{Q}x = 1$ by

0: $\neg\,\epsilon = x$

9: $\neg\,\mathrm{Q}x = 1 \quad \vee \quad \epsilon = x$

14: $\mathrm{Q}x = 1 \quad \vee \quad \mathrm{Parity}(\mathrm{Q}x) = 0$ by

7: $\mathrm{Q}x$ is a power of two

6: $\neg$ $\mathrm{Q}x$ is a power of two $\quad \vee \quad \mathrm{Q}x = 1 \quad \vee \quad \mathrm{Parity}(\mathrm{Q}x) = 0$

15: $\neg\, \mathrm{Parity}(\mathrm{Q}x) = 0 \quad \vee \quad \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x)$ by

8: $\mathrm{R}x < \mathrm{Q}x$

10: $\neg\, \mathrm{Parity}(\mathrm{Q}x) = 0 \quad \vee \quad \neg\, \mathrm{R}x < \mathrm{Q}x \quad \vee \quad \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x)$

16: $\neg$ $\mathrm{Half}(\mathrm{Q}x)$ is a power of two $\vee$ $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x)$ by

11: $(\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x)$

2: $\neg\, (\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x) \quad \vee$ $\neg$ $\mathrm{Half}(\mathrm{Q}x)$ is a power of two $\vee$ $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x)$

17: $\neg$ $\mathrm{Half}(\mathrm{Q}x)$ is a power of two $\vee$ $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$ by

11: $(\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x)$

3: $\neg\, (\mathrm{Half}(\mathrm{Q}x)) + (\mathrm{Half}(\mathrm{R}x)) = \mathrm{S}(\mathrm{Chop}x) \quad \vee$ $\neg$ $\mathrm{Half}(\mathrm{Q}x)$ is a power of two $\vee$ $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$

18: $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x)$ by

12: $\mathrm{Half}(\mathrm{Q}x)$ is a power of two

16: $\neg$ $\mathrm{Half}(\mathrm{Q}x)$ is a power of two $\vee$ $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x)$

19: $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$ by

12: $\mathrm{Half}(\mathrm{Q}x)$ is a power of two

17: $\neg$ $\mathrm{Half}(\mathrm{Q}x)$ is a power of two $\vee$ $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$

20: $\mathrm{Parity}(\mathrm{Q}x) = 0$ by

13: $\neg\, \mathrm{Q}x = 1$

14: $\mathrm{Q}x = 1 \quad \vee \quad \mathrm{Parity}(\mathrm{Q}x) = 0$

21: $\mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x)$ by

20: $\mathrm{Parity}(\mathrm{Q}x) = 0$

15: $\neg\, \mathrm{Parity}(\mathrm{Q}x) = 0 \quad \vee \quad \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x)$

22: $\mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x)$ by

21: $\mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x)$

18: $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x)$

23: $\mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$ by

21: $\mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x)$

19: $\neg\, \mathrm{Half}(\mathrm{R}x) < \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$

24: $\neg\, \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$ by

22: $\mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x)$

1: $\neg\, \mathrm{Q}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{Q}x) \quad \vee \quad \neg\, \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$

25: $QEA$ by

23: $\mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$

24: $\neg\, \mathrm{R}(\mathrm{Chop}x) = \mathrm{Half}(\mathrm{R}x)$