Proof of Theorem 233

The theorem to be proved is

$\text{Parity}\, q = 0 \quad \& \quad r < q \quad \rightarrow \quad \text{Half}\, r < \text{Half}\, q$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(\text{Parity}q) = (0)] \quad \& \quad [(r) < (q)] \quad \& \quad [\neg\, (\text{Half}r) < (\text{Half}q)]]$

Special cases of the hypothesis and previous results:

0: $\text{Parity}q = 0$ from H:q:r

1: $r < q$ from H:q:r

2: $\neg\, \text{Half}r < \text{Half}q$ from H:q:r

3: $\text{Half}q \leq \text{Half}r \quad \vee \quad \text{Half}r < \text{Half}q$ from 79;Half$q$;Half$r$

4: $\neg\, \text{Parity}q = 0 \quad \vee \quad 2 \cdot (\text{Half}q) = q$ from 224;q

5: $(\text{Half}q) + (\text{Half}q) = 2 \cdot (\text{Half}q)$ from 118;Half$q$

6: $\neg\, \text{Half}q \leq \text{Half}r \quad \vee \quad \neg\, \text{Half}q \leq \text{Half}r \quad \vee \quad (\text{Half}q) + (\text{Half}q) \leq (\text{Half}r) + (\text{Half}r)$ from 184;Half$q$;Half$r$;Half$q$;Half$r$

7: $2 \cdot (\text{Half}r) \leq r$ from 232;r

8: $(\text{Half}r) + (\text{Half}r) = 2 \cdot (\text{Half}r)$ from 118;Half$r$

9: $\neg\, q \leq (\text{Half}r) + (\text{Half}r) \quad \vee \quad \neg\, (\text{Half}r) + (\text{Half}r) \leq r \quad \vee \quad q \leq r$ from 73;q;$(\text{Half}r) + (\text{Half}r)$;$r$

10: $\neg\, r < q \quad \vee \quad \neg\, q \leq r$ from 80;r;q

Equality substitutions:

11: $\neg\, 2 \cdot (\text{Half}q) = q \quad \vee \quad \neg\, (\text{Half}q) + (\text{Half}q) = 2 \cdot (\text{Half}q) \quad \vee \quad (\text{Half}q) + (\text{Half}q) = q$

12: $\neg\, (\text{Half}r) + (\text{Half}r) = 2 \cdot (\text{Half}r) \quad \vee \quad (\text{Half}r) + (\text{Half}r) \leq r \quad \vee \quad \neg\, 2 \cdot (\text{Half}r) \leq r$

13: $\neg\, (\text{Half}q) + (\text{Half}q) = q \quad \vee \quad \neg\, (\text{Half}q) + (\text{Half}q) \leq (\text{Half}r) + (\text{Half}r) \quad \vee \quad q \leq (\text{Half}r) + (\text{Half}r)$

Inferences:

14: $2 \cdot (\text{Half}q) = q$ by

0: $\text{Parity}q = 0$

4: $\neg\, \text{Parity}q = 0 \quad \vee \quad 2 \cdot (\text{Half}q) = q$

15: $\neg\, q \leq r$ by

1: $r < q$

10: $\neg\, r < q \quad \vee \quad \neg\, q \leq r$

16: $\text{Half}q \leq \text{Half}r$ by

2: $\neg\, \text{Half}r < \text{Half}q$

3: $\text{Half}q \leq \text{Half}r \quad \vee \quad \text{Half}r < \text{Half}q$

17: $\neg\, 2 \cdot (\text{Half}q) = q \quad \vee \quad (\text{Half}q) + (\text{Half}q) = q$ by

5: $(\text{Half}q) + (\text{Half}q) = 2 \cdot (\text{Half}q)$

11: $\neg\, 2 \cdot (\text{Half}q) = q \quad \vee \quad \neg\, (\text{Half}q) + (\text{Half}q) = 2 \cdot (\text{Half}q) \quad \vee \quad (\text{Half}q) + (\text{Half}q) = q$

18: $\neg\, (\text{Half}r) + (\text{Half}r) = 2 \cdot (\text{Half}r) \quad \vee \quad (\text{Half}r) + (\text{Half}r) \leq r$ by

7: $2 \cdot (\text{Half}r) \leq r$

12: $\neg\, (\text{Half}r) + (\text{Half}r) = 2 \cdot (\text{Half}r) \quad \vee \quad (\text{Half}r) + (\text{Half}r) \leq r \quad \vee \quad \neg\, 2 \cdot (\text{Half}r) \leq r$

19: $(\text{Half}r) + (\text{Half}r) \leq r$ by

8: $(\text{Half}r) + (\text{Half}r) = 2 \cdot (\text{Half}r)$

18: $\neg\, (\text{Half}r) + (\text{Half}r) = 2 \cdot (\text{Half}r) \quad \vee \quad (\text{Half}r) + (\text{Half}r) \leq r$

20: $(\text{Half}q) + (\text{Half}q) = q$ by

14: $2 \cdot (\text{Half}q) = q$

17: $\neg\, 2 \cdot (\text{Half}q) = q \quad \vee \quad (\text{Half}q) + (\text{Half}q) = q$

21: $\neg\, q \leq (\text{Half}r) + (\text{Half}r) \quad \vee \quad \neg\, (\text{Half}r) + (\text{Half}r) \leq r$ by

15: $\neg\, q \leq r$

9: $\neg\, q \leq (\text{Half}r) + (\text{Half}r) \quad \vee \quad \neg\, (\text{Half}r) + (\text{Half}r) \leq r \quad \vee \quad q \leq r$

22: $(\text{Half}q) + (\text{Half}q) \leq (\text{Half}r) + (\text{Half}r)$ by

16: $\text{Half}q \leq \text{Half}r$

6: $\neg\, \text{Half}q \leq \text{Half}r \quad \vee \quad \neg\, \text{Half}q \leq \text{Half}r \quad \vee \quad (\text{Half}q) + (\text{Half}q) \leq (\text{Half}r) + (\text{Half}r)$

23: $\neg\, q \leq (\text{Half}r) + (\text{Half}r)$ by

19: $(\text{Half}r) + (\text{Half}r) \leq r$

21: $\neg\, q \leq (\text{Half}r) + (\text{Half}r) \quad \vee \quad \neg\, (\text{Half}r) + (\text{Half}r) \leq r$

24: $\neg\, (\text{Half}q) + (\text{Half}q) \leq (\text{Half}r) + (\text{Half}r) \quad \vee \quad q \leq (\text{Half}r) + (\text{Half}r)$ by

20: $(\text{Half}q) + (\text{Half}q) = q$

13: $\neg\, (\text{Half}q) + (\text{Half}q) = q \quad \vee \quad \neg\, (\text{Half}q) + (\text{Half}q) \leq (\text{Half}r) + (\text{Half}r) \quad \vee \quad q \leq (\text{Half}r) + (\text{Half}r)$

25: $q \leq (\text{Half}r) + (\text{Half}r)$ by

22: $(\text{Half}q) + (\text{Half}q) \leq (\text{Half}r) + (\text{Half}r)$

24: $\neg\ (\text{Half}q) + (\text{Half}q) \leq (\text{Half}r) + (\text{Half}r) \quad \vee \quad q \leq (\text{Half}r) + (\text{Half}r)$

26: QEA by

23: $\neg\ q \leq (\text{Half}r) + (\text{Half}r)$

25: $q \leq (\text{Half}r) + (\text{Half}r)$