Proof of Theorem 22i

The theorem to be proved is

$[x \neq 0 \quad \rightarrow \quad x = \mathrm{SP}x] \quad \rightarrow \quad [\mathrm{S}x \neq 0 \quad \rightarrow \quad \mathrm{S}x = \mathrm{SPS}x]$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) = (0) \quad \vee \quad (x) = (\mathrm{S}(\mathrm{P}x))] \quad \& \quad [\neg\, (\mathrm{S}x) = (0)] \quad \& \quad [\neg\, (\mathrm{S}x) = (\mathrm{S}(\mathrm{P}(\mathrm{S}x)))]]$

Special cases of the hypothesis and previous results:

0: $\neg\, \mathrm{S}(\mathrm{P}(\mathrm{S}x)) = \mathrm{S}x$ from H:x

1: $\mathrm{P}(\mathrm{S}x) = x$ from 16;x

Equality substitutions:

2: $\neg\, \mathrm{P}(\mathrm{S}x) = x \quad \vee \quad \mathrm{S}(\mathrm{P}(\mathrm{S}x)) = \mathrm{S}x \quad \vee \quad \neg\, \mathrm{S}(x) = \mathrm{S}x$

Inferences:

3: $\neg\, \mathrm{P}(\mathrm{S}x) = x$ by
0: $\neg\, \mathrm{S}(\mathrm{P}(\mathrm{S}x)) = \mathrm{S}x$
2: $\neg\, \mathrm{P}(\mathrm{S}x) = x \quad \vee \quad \mathrm{S}(\mathrm{P}(\mathrm{S}x)) = \mathrm{S}x$

4: QEA by
1: $\mathrm{P}(\mathrm{S}x) = x$
3: $\neg\, \mathrm{P}(\mathrm{S}x) = x$