Proof of Theorem 227

The theorem to be proved is

Half $x = 0 \quad \rightarrow \quad x = 0 \quad \vee \quad x = 1$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(\text{Half}x) = (0)] \quad \& \quad [\neg\, (x) = (0)] \quad \& \quad [\neg\, (x) = (1)]]$

Special cases of the hypothesis and previous results:

0: $\text{Half}x = 0$ from H:x

1: $\neg\, 0 = x$ from H:x

2: $\neg\, 1 = x$ from H:x

3: $x \leq (2 \cdot (\text{Half}x)) + 1$ from 226;x

4: $2 \cdot 0 = 0$ from 100;2

5: $0 + 1 = 1$ from 97;1

6: $\neg\, x \leq 1 \quad \vee \quad 0 = x \quad \vee \quad 1 = x$ from 200;x

Equality substitutions:

7: $\neg\, \text{Half}x = 0 \quad \vee \quad \neg\, x \leq (2 \cdot (\text{Half}x)) + 1 \quad \vee \quad x \leq (2 \cdot (0)) + 1$

8: $\neg\, 2 \cdot 0 = 0 \quad \vee \quad \neg\, x \leq (2 \cdot 0) + 1 \quad \vee \quad x \leq (0) + 1$

9: $\neg\, 0 + 1 = 1 \quad \vee \quad \neg\, x \leq 0 + 1 \quad \vee \quad x \leq 1$

Inferences:

10: $\neg\, x \leq (2 \cdot (\text{Half}x)) + 1 \quad \vee \quad x \leq (2 \cdot 0) + 1$ by
0: $\text{Half}x = 0$
7: $\neg\, \text{Half}x = 0 \quad \vee \quad \neg\, x \leq (2 \cdot (\text{Half}x)) + 1 \quad \vee \quad x \leq (2 \cdot 0) + 1$

11: $\neg\, x \leq 1 \quad \vee \quad 1 = x$ by
1: $\neg\, 0 = x$
6: $\neg\, x \leq 1 \quad \vee \quad 0 = x \quad \vee \quad 1 = x$

12: $\neg\, x \leq 1$ by
2: $\neg\, 1 = x$
11: $\neg\, x \leq 1 \quad \vee \quad 1 = x$

13: $x \leq (2 \cdot 0) + 1$ by

3: $x \leq (2 \cdot (\text{Half}x)) + 1$

10: $\neg\, x \leq (2 \cdot (\text{Half}x)) + 1 \quad \vee \quad x \leq (2 \cdot 0) + 1$

14: $\neg\, x \leq (2 \cdot 0) + 1 \quad \vee \quad x \leq 0 + 1$ by

4: $2 \cdot 0 = 0$

8: $\neg\, 2 \cdot 0 = 0 \quad \vee \quad \neg\, x \leq (2 \cdot 0) + 1 \quad \vee \quad x \leq 0 + 1$

15: $\neg\, x \leq 0 + 1 \quad \vee \quad x \leq 1$ by

5: $0 + 1 = 1$

9: $\neg\, 0 + 1 = 1 \quad \vee \quad \neg\, x \leq 0 + 1 \quad \vee \quad x \leq 1$

16: $\neg\, x \leq 0 + 1$ by

12: $\neg\, x \leq 1$

15: $\neg\, x \leq 0 + 1 \quad \vee \quad x \leq 1$

17: $x \leq 0 + 1$ by

13: $x \leq (2 \cdot 0) + 1$

14: $\neg\, x \leq (2 \cdot 0) + 1 \quad \vee \quad x \leq 0 + 1$

18: QEA by

16: $\neg\, x \leq 0 + 1$

17: $x \leq 0 + 1$