Proof of Theorem 204

The theorem to be proved is

$x \oplus y = \epsilon \quad \rightarrow \quad x = \epsilon \quad \& \quad y = \epsilon$ ⋆

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x \oplus y) = (\epsilon)] \quad \& \quad [\neg\, (x) = (\epsilon) \quad \vee \quad \neg\, (y) = (\epsilon)]]$

Special cases of the hypothesis and previous results:

0: $x \oplus y = \epsilon$ from H:x:y

1: $\neg\, \epsilon = x \quad \vee \quad \neg\, \epsilon = y$ from H:x:y

2: $(\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}(x \oplus y)$ from 180;x;y

3: $\mathrm{Q}\epsilon = 1$ from 189

4: $\neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1 \quad \vee \quad \mathrm{Q}x = 1$ from 202;Qx;Qy

5: $\neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1 \quad \vee \quad \mathrm{Q}y = 1$ from 202;Qx;Qy

6: $\neg\, \mathrm{Q}x = 1 \quad \vee \quad \epsilon = x$ from 203;x

7: $\neg\, \mathrm{Q}y = 1 \quad \vee \quad \epsilon = y$ from 203;y

Equality substitutions:

8: $\neg\, x \oplus y = \epsilon \quad \vee \quad \neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}(x \oplus y) \quad \vee \quad (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}(\epsilon)$

9: $\neg\, \mathrm{Q}\epsilon = 1 \quad \vee \quad \neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon \quad \vee \quad (\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1$

Inferences:

10: $\neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}(x \oplus y) \quad \vee \quad (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon$ by
0: $x \oplus y = \epsilon$
8: $\neg\, x \oplus y = \epsilon \quad \vee \quad \neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}(x \oplus y) \quad \vee \quad (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon$

11: $(\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon$ by
2: $(\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}(x \oplus y)$
10: $\neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}(x \oplus y) \quad \vee \quad (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon$

12: $\neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon \quad \vee \quad (\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1$ by
3: $\mathrm{Q}\epsilon = 1$
9: $\neg\, \mathrm{Q}\epsilon = 1 \quad \vee \quad \neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon \quad \vee \quad (\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1$

13: $(\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1$ by

11: $(\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon$

12: $\neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = \mathrm{Q}\epsilon \quad \vee \quad (\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1$

14: $\mathrm{Q}x = 1$ by

13: $(\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1$

4: $\neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1 \quad \vee \quad \mathrm{Q}x = 1$

15: $\mathrm{Q}y = 1$ by

13: $(\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1$

5: $\neg\, (\mathrm{Q}x) \cdot (\mathrm{Q}y) = 1 \quad \vee \quad \mathrm{Q}y = 1$

16: $\epsilon = x$ by

14: $\mathrm{Q}x = 1$

6: $\neg\, \mathrm{Q}x = 1 \quad \vee \quad \epsilon = x$

17: $\epsilon = y$ by

15: $\mathrm{Q}y = 1$

7: $\neg\, \mathrm{Q}y = 1 \quad \vee \quad \epsilon = y$

18: $\neg\, \epsilon = y$ by

16: $\epsilon = x$

1: $\neg\, \epsilon = x \quad \vee \quad \neg\, \epsilon = y$

19: QEA by

17: $\epsilon = y$

18: $\neg\, \epsilon = y$