Proof of Theorem 200

The theorem to be proved is

$x \le 1 \quad \rightarrow \quad x = 0 \quad \vee \quad x = 1$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) \le (1)] \quad \& \quad [\neg\, (x) = (0)] \quad \& \quad [\neg\, (x) = (1)]]$

Special cases of the hypothesis and previous results:

0: $x \le 1$ from H:x

1: $\neg\, 0 = x$ from H:x

2: $\neg\, 1 = x$ from H:x

3: $\mathrm{S}0 = 1$ from 115

4: $\neg\, x \le 1 \quad \vee \quad x + w = 1$ from 167;x;1:w

5: $0 = w \quad \vee \quad \mathrm{S}(\mathrm{P}w) = w$ from 22;w

6: $x + 0 = x$ from 12;x;Pw

7: $\mathrm{S}(x + (\mathrm{P}w)) = x + (\mathrm{S}(\mathrm{P}w))$ from 12;x;Pw

8: $\neg\, \mathrm{S}(x + (\mathrm{P}w)) = \mathrm{S}0 \quad \vee \quad x + (\mathrm{P}w) = 0$ from 4;$x + (\mathrm{P}w)$;0

9: $\neg\, x + (\mathrm{P}w) = 0 \quad \vee \quad 0 = x$ from 15;x;Pw

Equality substitutions:

10: $\neg\, \mathrm{S}0 = 1 \quad \vee \quad x + w = \mathrm{S}0 \quad \vee \quad \neg\, x + w = 1$

11: $\neg\, x + w = 1 \quad \vee \quad \neg\, x + w = x \quad \vee \quad 1 = x$

12: $\neg\, 0 = w \quad \vee \quad \neg\, x + 0 = x \quad \vee \quad x + w = x$

13: $\neg\, \mathrm{S}(\mathrm{P}w) = w \quad \vee \quad x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0 \quad \vee \quad \neg\, x + (w) = \mathrm{S}0$

14: $\neg\, \mathrm{S}(x + (\mathrm{P}w)) = x + (\mathrm{S}(\mathrm{P}w)) \quad \vee \quad \mathrm{S}(x + (\mathrm{P}w)) = \mathrm{S}0 \quad \vee \quad \neg\, x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0$

Inferences:

15: $x + w = 1$ by

0: $x \le 1$

4: $\neg\, x \le 1 \quad \vee \quad x + w = 1$

16: $\neg\, x + (\mathrm{P}w) = 0$ by

1: $\neg\, 0 = x$

9: $\neg\, x + (\mathrm{P}w) = 0 \quad \vee \quad 0 = x$

17: $\neg\, x + w = 1 \quad \vee \quad \neg\, x + w = x$ by

2: $\neg\, 1 = x$

11: $\neg\, x + w = 1 \quad \vee \quad \neg\, x + w = x \quad \vee \quad 1 = x$

18: $x + w = \mathrm{S}0 \quad \vee \quad \neg\, x + w = 1$ by

3: $\mathrm{S}0 = 1$

10: $\neg\, \mathrm{S}0 = 1 \quad \vee \quad x + w = \mathrm{S}0 \quad \vee \quad \neg\, x + w = 1$

19: $\neg\, 0 = w \quad \vee \quad x + w = x$ by

6: $x + 0 = x$

12: $\neg\, 0 = w \quad \vee \quad \neg\, x + 0 = x \quad \vee \quad x + w = x$

20: $\mathrm{S}(x + (\mathrm{P}w)) = \mathrm{S}0 \quad \vee \quad \neg\, x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0$ by

7: $\mathrm{S}(x + (\mathrm{P}w)) = x + (\mathrm{S}(\mathrm{P}w))$

14: $\neg\, \mathrm{S}(x + (\mathrm{P}w)) = x + (\mathrm{S}(\mathrm{P}w)) \quad \vee \quad \mathrm{S}(x + (\mathrm{P}w)) = \mathrm{S}0 \quad \vee \quad \neg\, x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0$

21: $\neg\, x + w = x$ by

15: $x + w = 1$

17: $\neg\, x + w = 1 \quad \vee \quad \neg\, x + w = x$

22: $x + w = \mathrm{S}0$ by

15: $x + w = 1$

18: $x + w = \mathrm{S}0 \quad \vee \quad \neg\, x + w = 1$

23: $\neg\, \mathrm{S}(x + (\mathrm{P}w)) = \mathrm{S}0$ by

16: $\neg\, x + (\mathrm{P}w) = 0$

8: $\neg\, \mathrm{S}(x + (\mathrm{P}w)) = \mathrm{S}0 \quad \vee \quad x + (\mathrm{P}w) = 0$

24: $\neg\, 0 = w$ by

21: $\neg\, x + w = x$

19: $\neg\, 0 = w \quad \vee \quad x + w = x$

25: $\neg\, \mathrm{S}(\mathrm{P}w) = w \quad \vee \quad x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0$ by

22: $x + w = \mathrm{S}0$

13: $\neg\, \mathrm{S}(\mathrm{P}w) = w \quad \vee \quad x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0 \quad \vee \quad \neg\, x + w = \mathrm{S}0$

26: $\neg\, x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0$ by

23: $\neg\, \mathrm{S}(x + (\mathrm{P}w)) = \mathrm{S}0$

20: $\mathrm{S}(x + (\mathrm{P}w)) = \mathrm{S}0 \quad \vee \quad \neg\, x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0$

27: $\mathrm{S}(\mathrm{P}w) = w$ by

24: $\neg\, 0 = w$

5: $0 = w \quad \vee \quad \mathrm{S}(\mathrm{P}w) = w$

28: $\neg\, \mathrm{S}(\mathrm{P}w) = w$ by

26: $\neg\, x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0$

25: $\neg\, \mathrm{S}(\mathrm{P}w) = w \quad \vee \quad x + (\mathrm{S}(\mathrm{P}w)) = \mathrm{S}0$

29: QEA by

27: $\mathrm{S}(\mathrm{P}w) = w$

28: $\neg\, \mathrm{S}(\mathrm{P}w) = w$