

Proof of Theorem 188

The theorem to be proved is

$$\underline{0} \neq \epsilon \quad \& \quad \underline{1} \neq \epsilon \quad \& \quad \underline{0} \neq \underline{1} \tag{*}$$

Suppose the theorem does not hold. Then, with the variables held fixed,

$$(H) \quad [[(\underline{0}) = (\epsilon) \quad \vee \quad (\underline{1}) = (\epsilon) \quad \vee \quad (\underline{0}) = (\underline{1})]]$$

Special cases of the hypothesis and previous results:

- 0: $\underline{0} = \epsilon \quad \vee \quad \underline{1} = \epsilon \quad \vee \quad \underline{1} = \underline{0}$ from H
- 1: $S(S0) = 2$ from [116](#)
- 2: $S0 = 1$ from [115](#)
- 3: $\epsilon = 0$ from [185](#)
- 4: $\underline{0} = 1$ from [186](#)
- 5: $\underline{1} = 2$ from [187](#)
- 6: $\neg S0 = 0$ from [3](#);0
- 7: $\neg S1 = 0$ from [3](#);1
- 8: $\neg S1 = S0 \quad \vee \quad 1 = 0$ from [4](#);0;1

Equality substitutions:

- 9: $\neg \underline{0} = \epsilon \quad \vee \quad \neg \underline{0} = 1 \quad \vee \quad \epsilon = 1$
- 10: $\neg \underline{1} = \epsilon \quad \vee \quad \neg \underline{1} = 2 \quad \vee \quad \epsilon = 2$
- 11: $\neg \underline{1} = \underline{0} \quad \vee \quad \neg \underline{1} = 2 \quad \vee \quad \underline{0} = 2$
- 12: $\neg S0 = 1 \quad \vee \quad \neg S(S0) = 2 \quad \vee \quad S(1) = 2$
- 13: $\neg S0 = 1 \quad \vee \quad S0 = 0 \quad \vee \quad \neg 1 = 0$
- 14: $\neg S0 = 1 \quad \vee \quad S1 = S0 \quad \vee \quad \neg S1 = 1$
- 15: $\neg \epsilon = 0 \quad \vee \quad \neg \epsilon = 1 \quad \vee \quad 0 = 1$
- 16: $\neg \epsilon = 0 \quad \vee \quad \neg \epsilon = 2 \quad \vee \quad 0 = 2$
- 17: $\neg \underline{0} = 1 \quad \vee \quad \neg \underline{0} = 2 \quad \vee \quad 1 = 2$

$$18: \neg S1 = 2 \vee S1 = 0 \vee \neg 2 = 0$$

$$19: \neg S1 = 2 \vee S1 = 1 \vee \neg 2 = 1$$

Inferences:

$$20: \neg S0 = 1 \vee S1 = 2 \quad \text{by}$$

$$1: S(S0) = 2$$

$$12: \neg S0 = 1 \vee \neg S(S0) = 2 \vee S1 = 2$$

$$21: S0 = 0 \vee \neg 1 = 0 \quad \text{by}$$

$$2: S0 = 1$$

$$13: \neg S0 = 1 \vee S0 = 0 \vee \neg 1 = 0$$

$$22: S1 = S0 \vee \neg S1 = 1 \quad \text{by}$$

$$2: S0 = 1$$

$$14: \neg S0 = 1 \vee S1 = S0 \vee \neg S1 = 1$$

$$23: S1 = 2 \quad \text{by}$$

$$2: S0 = 1$$

$$20: \neg S0 = 1 \vee S1 = 2$$

$$24: \neg \epsilon = 1 \vee 1 = 0 \quad \text{by}$$

$$3: \epsilon = 0$$

$$15: \neg \epsilon = 0 \vee \neg \epsilon = 1 \vee 1 = 0$$

$$25: \neg \epsilon = 2 \vee 2 = 0 \quad \text{by}$$

$$3: \epsilon = 0$$

$$16: \neg \epsilon = 0 \vee \neg \epsilon = 2 \vee 2 = 0$$

$$26: \neg \underline{0} = \epsilon \vee \epsilon = 1 \quad \text{by}$$

$$4: \underline{0} = 1$$

$$9: \neg \underline{0} = \epsilon \vee \neg \underline{0} = 1 \vee \epsilon = 1$$

$$27: \neg \underline{0} = 2 \vee 2 = 1 \quad \text{by}$$

$$4: \underline{0} = 1$$

$$17: \neg \underline{0} = 1 \vee \neg \underline{0} = 2 \vee 2 = 1$$

$$28: \neg \underline{1} = \epsilon \vee \epsilon = 2 \quad \text{by}$$

$$5: \underline{1} = 2$$

$$10: \neg \underline{1} = \epsilon \vee \neg \underline{1} = 2 \vee \epsilon = 2$$

- 29: $\neg \underline{1} = \underline{0} \vee \underline{0} = 2$ by
5: $\underline{1} = 2$
11: $\neg \underline{1} = \underline{0} \vee \neg \underline{1} = 2 \vee \underline{0} = 2$
- 30: $\neg 1 = 0$ by
6: $\neg S0 = 0$
21: $S0 = 0 \vee \neg 1 = 0$
- 31: $\neg S1 = 2 \vee \neg 2 = 0$ by
7: $\neg S1 = 0$
18: $\neg S1 = 2 \vee S1 = 0 \vee \neg 2 = 0$
- 32: $S1 = 1 \vee \neg 2 = 1$ by
23: $S1 = 2$
19: $\neg S1 = 2 \vee S1 = 1 \vee \neg 2 = 1$
- 33: $\neg 2 = 0$ by
23: $S1 = 2$
31: $\neg S1 = 2 \vee \neg 2 = 0$
- 34: $\neg S1 = S0$ by
30: $\neg 1 = 0$
8: $\neg S1 = S0 \vee 1 = 0$
- 35: $\neg \epsilon = 1$ by
30: $\neg 1 = 0$
24: $\neg \epsilon = 1 \vee 1 = 0$
- 36: $\neg \epsilon = 2$ by
33: $\neg 2 = 0$
25: $\neg \epsilon = 2 \vee 2 = 0$
- 37: $\neg S1 = 1$ by
34: $\neg S1 = S0$
22: $S1 = S0 \vee \neg S1 = 1$
- 38: $\neg \underline{0} = \epsilon$ by
35: $\neg \epsilon = 1$
26: $\neg \underline{0} = \epsilon \vee \epsilon = 1$
- 39: $\neg \underline{1} = \epsilon$ by
36: $\neg \epsilon = 2$
28: $\neg \underline{1} = \epsilon \vee \epsilon = 2$

- 40: $\neg 2 = 1$ by
 37: $\neg S1 = 1$
 32: $S1 = 1 \vee \neg 2 = 1$
- 41: $\underline{1} = \epsilon \vee \underline{1} = \underline{0}$ by
 38: $\neg \underline{0} = \epsilon$
 0: $\underline{0} = \epsilon \vee \underline{1} = \epsilon \vee \underline{1} = \underline{0}$
- 42: $\underline{1} = \underline{0}$ by
 39: $\neg \underline{1} = \epsilon$
 41: $\underline{1} = \epsilon \vee \underline{1} = \underline{0}$
- 43: $\neg \underline{0} = 2$ by
 40: $\neg 2 = 1$
 27: $\neg \underline{0} = 2 \vee 2 = 1$
- 44: $\underline{0} = 2$ by
 42: $\underline{1} = \underline{0}$
 29: $\neg \underline{1} = \underline{0} \vee \underline{0} = 2$
- 45: *QEA* by
 43: $\neg \underline{0} = 2$
 44: $\underline{0} = 2$