

Proof of Theorem 184

The theorem to be proved is

$$x_1 \leq y_1 \ \& \ x_2 \leq y_2 \ \rightarrow \ x_1 + x_2 \leq y_1 + y_2$$

Suppose the theorem does not hold. Then, with the variables held fixed,

$$(H) \quad [[(x_1) \leq (y_1)] \ \& \ [(x_2) \leq (y_2)] \ \& \ [\neg (x_1 + x_2) \leq (y_1 + y_2)]]$$

Special cases of the hypothesis and previous results:

- 0: $x_1 \leq y_1$ from H: $x_1:y_1:x_2:y_2$
- 1: $x_2 \leq y_2$ from H: $x_1:y_1:x_2:y_2$
- 2: $\neg x_1 + x_2 \leq y_1 + y_2$ from H: $x_1:y_1:x_2:y_2$
- 3: $\neg x_1 \leq y_1 \ \vee \ x_1 + w_1 = y_1$ from [167](#); $x_1;y_1:w_1$
- 4: $\neg x_2 \leq y_2 \ \vee \ x_2 + w_2 = y_2$ from [167](#); $x_2;y_2:w_2$
- 5: $x_1 + (w_1 + (x_2 + w_2)) = (x_1 + w_1) + (x_2 + w_2)$ from [72](#); $x_1;w_1;x_2 + w_2$
- 6: $w_1 + (x_2 + w_2) = (w_1 + x_2) + w_2$ from [72](#); $w_1;x_2;w_2$
- 7: $x_2 + w_1 = w_1 + x_2$ from [98](#); $w_1;x_2$
- 8: $x_2 + (w_1 + w_2) = (x_2 + w_1) + w_2$ from [72](#); $x_2;w_1;w_2$
- 9: $x_1 + (x_2 + (w_1 + w_2)) = (x_1 + x_2) + (w_1 + w_2)$ from [72](#); $x_1;x_2;w_1 + w_2$
- 10: $x_1 + x_2 \leq (x_1 + x_2) + (w_1 + w_2)$ from [71](#); $x_1 + x_2;w_1 + w_2$

Equality substitutions:

- 11: $\neg x_2 + w_2 = y_2 \ \vee \ \neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2)$
 $\vee \ x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (y_2)$
- 12: $\neg w_1 + (x_2 + w_2) = (w_1 + x_2) + w_2 \ \vee \ \neg x_1 + (w_1 + (x_2 + w_2)) = (x_1 + w_1) + (x_2 + w_2)$
 $\vee \ x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2)$
- 13: $\neg x_2 + w_1 = w_1 + x_2 \ \vee \ \neg x_2 + (w_1 + w_2) = (x_2 + w_1) + w_2 \ \vee \ x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2$
- 14: $\neg x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2 \ \vee \ \neg x_1 + (x_2 + (w_1 + w_2)) = (x_1 + x_2) + (w_1 + w_2)$
 $\vee \ x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2)$
- 15: $\neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2) \ \vee \ \neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + y_2$
 $\vee \ (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$

$$16: \neg (x_1 + x_2) + (w_1 + w_2) = y_1 + y_2 \quad \vee \quad \neg x_1 + x_2 \leq (x_1 + x_2) + (w_1 + w_2)$$

$$\vee x_1 + x_2 \leq y_1 + y_2$$

$$17: \neg y_1 = x_1 + w_1 \quad \vee \quad (x_1 + x_2) + (w_1 + w_2) = (y_1) + y_2 \quad \vee \quad \neg (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$$

Inferences:

$$18: x_1 + w_1 = y_1 \quad \text{by}$$

$$0: x_1 \leq y_1$$

$$3: \neg x_1 \leq y_1 \quad \vee \quad x_1 + w_1 = y_1$$

$$19: x_2 + w_2 = y_2 \quad \text{by}$$

$$1: x_2 \leq y_2$$

$$4: \neg x_2 \leq y_2 \quad \vee \quad x_2 + w_2 = y_2$$

$$20: \neg (x_1 + x_2) + (w_1 + w_2) = y_1 + y_2 \quad \vee \quad \neg x_1 + x_2 \leq (x_1 + x_2) + (w_1 + w_2) \quad \text{by}$$

$$2: \neg x_1 + x_2 \leq y_1 + y_2$$

$$16: \neg (x_1 + x_2) + (w_1 + w_2) = y_1 + y_2 \quad \vee \quad \neg x_1 + x_2 \leq (x_1 + x_2) + (w_1 + w_2)$$

$$\vee x_1 + x_2 \leq y_1 + y_2$$

$$21: \neg w_1 + (x_2 + w_2) = (w_1 + x_2) + w_2 \quad \vee \quad x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2)$$

by

$$5: x_1 + (w_1 + (x_2 + w_2)) = (x_1 + w_1) + (x_2 + w_2)$$

$$12: \neg w_1 + (x_2 + w_2) = (w_1 + x_2) + w_2 \quad \vee \quad \neg x_1 + (w_1 + (x_2 + w_2)) = (x_1 + w_1) + (x_2 + w_2)$$

$$\vee x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2)$$

$$22: x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2) \quad \text{by}$$

$$6: w_1 + (x_2 + w_2) = (w_1 + x_2) + w_2$$

$$21: \neg w_1 + (x_2 + w_2) = (w_1 + x_2) + w_2 \quad \vee \quad x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2)$$

$$23: \neg x_2 + (w_1 + w_2) = (x_2 + w_1) + w_2 \quad \vee \quad x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2 \quad \text{by}$$

$$7: x_2 + w_1 = w_1 + x_2$$

$$13: \neg x_2 + w_1 = w_1 + x_2 \quad \vee \quad \neg x_2 + (w_1 + w_2) = (x_2 + w_1) + w_2 \quad \vee \quad x_2 + (w_1 + w_2) =$$

$$(w_1 + x_2) + w_2$$

$$24: x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2 \quad \text{by}$$

$$8: x_2 + (w_1 + w_2) = (x_2 + w_1) + w_2$$

$$23: \neg x_2 + (w_1 + w_2) = (x_2 + w_1) + w_2 \quad \vee \quad x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2$$

$$25: \neg x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2 \quad \vee \quad x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2)$$

by

9: $x_1 + (x_2 + (w_1 + w_2)) = (x_1 + x_2) + (w_1 + w_2)$
 14: $\neg x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2 \quad \vee \quad \neg x_1 + (x_2 + (w_1 + w_2)) = (x_1 + x_2) + (w_1 + w_2)$
 $\vee \quad x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2)$
 26: $\neg (x_1 + x_2) + (w_1 + w_2) = y_1 + y_2$ by
 10: $x_1 + x_2 \leq (x_1 + x_2) + (w_1 + w_2)$
 20: $\neg (x_1 + x_2) + (w_1 + w_2) = y_1 + y_2 \quad \vee \quad \neg x_1 + x_2 \leq (x_1 + x_2) + (w_1 + w_2)$
 27: $(x_1 + x_2) + (w_1 + w_2) = y_1 + y_2 \quad \vee \quad \neg (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$ by
 18: $x_1 + w_1 = y_1$
 17: $\neg x_1 + w_1 = y_1 \quad \vee \quad (x_1 + x_2) + (w_1 + w_2) = y_1 + y_2 \quad \vee \quad \neg (x_1 + x_2) + (w_1 + w_2) =$
 $(x_1 + w_1) + y_2$
 28: $\neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2) \quad \vee \quad x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + y_2$
 by
 19: $x_2 + w_2 = y_2$
 11: $\neg x_2 + w_2 = y_2 \quad \vee \quad \neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2)$
 $\vee \quad x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + y_2$
 29: $x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + y_2$ by
 22: $x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2)$
 28: $\neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + (x_2 + w_2) \quad \vee \quad x_1 + ((w_1 + x_2) + w_2) =$
 $(x_1 + w_1) + y_2$
 30: $x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2)$ by
 24: $x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2$
 25: $\neg x_2 + (w_1 + w_2) = (w_1 + x_2) + w_2 \quad \vee \quad x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2)$
 31: $\neg (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$ by
 26: $\neg (x_1 + x_2) + (w_1 + w_2) = y_1 + y_2$
 27: $(x_1 + x_2) + (w_1 + w_2) = y_1 + y_2 \quad \vee \quad \neg (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$
 32: $\neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2) \quad \vee \quad (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$
 by
 29: $x_1 + ((w_1 + x_2) + w_2) = (x_1 + w_1) + y_2$
 15: $\neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2) \quad \vee \quad \neg x_1 + ((w_1 + x_2) + w_2) =$
 $(x_1 + w_1) + y_2 \quad \vee \quad (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$
 33: $(x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$ by
 30: $x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2)$

$$32: \neg x_1 + ((w_1 + x_2) + w_2) = (x_1 + x_2) + (w_1 + w_2) \quad \vee \quad (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$$

34: *QEA* by

$$31: \neg (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$$

$$33: (x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + y_2$$