Proof of Theorem 181

The theorem to be proved is

$Q\big(x \oplus (y \oplus z)\big) = Q\big((x \oplus y) \oplus z\big)$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\ (Q(x \oplus (y \oplus z))) = (Q((x \oplus y) \oplus z))]]$

Special cases of the hypothesis and previous results:

0: $\neg\ Q(x \oplus (y \oplus z)) = Q((x \oplus y) \oplus z)$ from H:x:y:z

1: $(Qy) \cdot (Qz) = Q(y \oplus z)$ from 180;y;z

2: $(Qx) \cdot (Qy) = Q(x \oplus y)$ from 180;x;y

3: $(Qx) \cdot (Q(y \oplus z)) = Q(x \oplus (y \oplus z))$ from 180;x;$y \oplus z$

4: $(Q(x \oplus y)) \cdot (Qz) = Q((x \oplus y) \oplus z)$ from 180;$x \oplus y$;z

5: $(Qx) \cdot ((Qy) \cdot (Qz)) = ((Qx) \cdot (Qy)) \cdot (Qz)$ from 102;Qx;Qy;Qz

Equality substitutions:

6: $\neg\ (Qy) \cdot (Qz) = Q(y \oplus z) \quad \vee \quad \neg\ (Qx) \cdot ((Qy) \cdot (Qz)) = ((Qx) \cdot (Qy)) \cdot (Qz)$
$\vee \quad (Qx) \cdot (Q(y \oplus z)) = ((Qx) \cdot (Qy)) \cdot (Qz)$

7: $\neg\ (Qx) \cdot (Qy) = Q(x \oplus y) \quad \vee \quad ((Qx) \cdot (Qy)) \cdot (Qz) = Q((x \oplus y) \oplus z) \quad \vee$
$\neg\ (Q(x \oplus y)) \cdot (Qz) = Q((x \oplus y) \oplus z)$

8: $\neg\ (Qx) \cdot (Q(y \oplus z)) = Q(x \oplus (y \oplus z)) \quad \vee \quad \neg\ (Qx) \cdot (Q(y \oplus z)) = Q((x \oplus y) \oplus z)$
$\vee \quad Q(x \oplus (y \oplus z)) = Q((x \oplus y) \oplus z)$

9: $\neg\ ((Qx) \cdot (Qy)) \cdot (Qz) = (Qx) \cdot (Q(y \oplus z)) \quad \vee \quad \neg\ ((Qx) \cdot (Qy)) \cdot (Qz) = Q((x \oplus y) \oplus z)$
$\vee \quad (Qx) \cdot (Q(y \oplus z)) = Q((x \oplus y) \oplus z)$

Inferences:

10: $\neg\ (Qx) \cdot (Q(y \oplus z)) = Q(x \oplus (y \oplus z)) \quad \vee \quad \neg\ (Qx) \cdot (Q(y \oplus z)) = Q((x \oplus y) \oplus z)$ by

0: $\neg\ Q(x \oplus (y \oplus z)) = Q((x \oplus y) \oplus z)$

8: $\neg\ (Qx) \cdot (Q(y \oplus z)) = Q(x \oplus (y \oplus z)) \quad \vee \quad \neg\ (Qx) \cdot (Q(y \oplus z)) = Q((x \oplus y) \oplus z)$
$\vee \quad Q(x \oplus (y \oplus z)) = Q((x \oplus y) \oplus z)$

11: $\neg\, (\mathrm{Q}x)\cdot((\mathrm{Q}y)\cdot(\mathrm{Q}z)) = ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) \quad \vee \quad ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z))$ by

1: $(\mathrm{Q}y)\cdot(\mathrm{Q}z) = \mathrm{Q}(y\oplus z)$

6: $\neg\, (\mathrm{Q}y)\cdot(\mathrm{Q}z) = \mathrm{Q}(y\oplus z) \quad \vee \quad \neg\, (\mathrm{Q}x)\cdot((\mathrm{Q}y)\cdot(\mathrm{Q}z)) = ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) \quad \vee \quad ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z))$

12: $((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z) \quad \vee \quad \neg\, (\mathrm{Q}(x\oplus y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z)$ by

2: $(\mathrm{Q}x)\cdot(\mathrm{Q}y) = \mathrm{Q}(x\oplus y)$

7: $\neg\, (\mathrm{Q}x)\cdot(\mathrm{Q}y) = \mathrm{Q}(x\oplus y) \quad \vee \quad ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z) \quad \vee \quad \neg\, (\mathrm{Q}(x\oplus y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z)$

13: $\neg\, (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) = \mathrm{Q}((x\oplus y)\oplus z)$ by

3: $(\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) = \mathrm{Q}(x\oplus(y\oplus z))$

10: $\neg\, (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) = \mathrm{Q}(x\oplus(y\oplus z)) \quad \vee \quad \neg\, (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) = \mathrm{Q}((x\oplus y)\oplus z)$

14: $((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z)$ by

4: $\mathrm{Q}(x\oplus y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z)$

12: $((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z) \quad \vee \quad \neg\, (\mathrm{Q}(x\oplus y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z)$

15: $((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z))$ by

5: $(\mathrm{Q}x)\cdot((\mathrm{Q}y)\cdot(\mathrm{Q}z)) = ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z)$

11: $\neg\, (\mathrm{Q}x)\cdot((\mathrm{Q}y)\cdot(\mathrm{Q}z)) = ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) \quad \vee \quad ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z))$

16: $\neg\, ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) \quad \vee \quad \neg\, ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z)$ by

13: $\neg\, (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) = \mathrm{Q}((x\oplus y)\oplus z)$

9: $\neg\, ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) \quad \vee \quad \neg\, ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z) \quad \vee \quad (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) = \mathrm{Q}((x\oplus y)\oplus z)$

17: $\neg\, ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z))$ by

14: $((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z)$

16: $\neg\, ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z)) \quad \vee \quad \neg\, ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = \mathrm{Q}((x\oplus y)\oplus z)$

18: QEA by

15: $((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z))$

17: $\neg\, ((\mathrm{Q}x)\cdot(\mathrm{Q}y))\cdot(\mathrm{Q}z) = (\mathrm{Q}x)\cdot(\mathrm{Q}(y\oplus z))$