Proof of Theorem 175

The theorem to be proved is

$r < q \quad \& \quad r' < q' \quad \rightarrow \quad r \cdot q' + r' < q \cdot q'$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(r) < (q)] \quad \& \quad [(r') < (q')] \quad \& \quad [\neg ((r \cdot q') + r') < (q \cdot q')]]$

Special cases of the hypothesis and previous results:

0: $r < q$ from H:r:q:r':q'

1: $r' < q'$ from H:r:q:r':q'

2: $\neg (r \cdot q') + r' < q \cdot q'$ from H:r:q:r':q'

3: $\neg r < q \quad \vee \quad r + w = q$ from 162;r;q:w

4: $\neg r < q \quad \vee \quad \neg 0 = w$ from 162;r;q:w

5: $\neg r' < q' \quad \vee \quad r' + w' = q'$ from 162;r';q':w'

6: $(r \cdot r') + (r \cdot w') = r \cdot (r' + w')$ from 101;r;r';w'

7: $(r \cdot (r' + w')) + (w \cdot (r' + w')) = (r + w) \cdot (r' + w')$ from 106;r;w;$r' + w'$

8: $0 = w \quad \vee \quad \neg r' < q' \quad \vee \quad r' < w \cdot q'$ from 173;w;r';q'

9: $\neg r' < w \cdot q' \quad \vee \quad ((r \cdot r') + (r \cdot w')) + r' < ((r \cdot r') + (r \cdot w')) + (w \cdot q')$ from 174;r';$w \cdot q'$;$r \cdot r') + (r \cdot w'$

Equality substitutions:

10: $\neg r' + w' = q' \quad \vee \quad \neg (r \cdot r') + (r \cdot w') = r \cdot (r' + w') \quad \vee \quad (r \cdot r') + (r \cdot w') = r \cdot (q')$

11: $\neg r' + w' = q' \quad \vee \quad \neg (r \cdot (r' + w')) + (w \cdot (r' + w')) = (r + w) \cdot (r' + w') \quad \vee \quad (r \cdot (q')) + (w \cdot (q')) = (r + w) \cdot (q')$

12: $\neg (r \cdot r') + (r \cdot w') = r \cdot q' \quad \vee \quad \neg ((r \cdot r') + (r \cdot w')) + r' < ((r \cdot r') + (r \cdot w')) + (w \cdot q') \quad \vee \quad (r \cdot q') + r' < (r \cdot q') + (w \cdot q')$

13: $\neg (r \cdot q') + (w \cdot q') = (r + w) \cdot q' \quad \vee \quad \neg (r \cdot q') + r' < (r \cdot q') + (w \cdot q') \quad \vee \quad (r \cdot q') + r' < (r + w) \cdot q'$

14: $\neg q = r + w \quad \vee \quad (r \cdot q') + r' < (q) \cdot q' \quad \vee \quad \neg (r \cdot q') + r' < (r + w) \cdot q'$

Inferences:

15: $r+w=q$ by

0: $r<q$

3: $\neg\, r<q \quad\lor\quad r+w=q$

16: $\neg\, 0=w$ by

0: $r<q$

4: $\neg\, r<q \quad\lor\quad \neg\, 0=w$

17: $r'+w'=q'$ by

1: $r'<q'$

5: $\neg\, r'<q' \quad\lor\quad r'+w'=q'$

18: $0=w \quad\lor\quad r'<w\cdot q'$ by

1: $r'<q'$

8: $0=w \quad\lor\quad \neg\, r'<q' \quad\lor\quad r'<w\cdot q'$

19: $\neg\, r+w=q \quad\lor\quad \neg\,(r\cdot q')+r'<(r+w)\cdot q'$ by

2: $\neg\,(r\cdot q')+r'<q\cdot q'$

14: $\neg\, r+w=q \quad\lor\quad (r\cdot q')+r'<q\cdot q' \quad\lor\quad \neg\,(r\cdot q')+r'<(r+w)\cdot q'$

20: $\neg\, r'+w'=q' \quad\lor\quad (r\cdot r')+(r\cdot w')=r\cdot q'$ by

6: $(r\cdot r')+(r\cdot w')=r\cdot(r'+w')$

10: $\neg\, r'+w'=q' \quad\lor\quad \neg\,(r\cdot r')+(r\cdot w')=r\cdot(r'+w') \quad\lor\quad (r\cdot r')+(r\cdot w')=r\cdot q'$

21: $\neg\, r'+w'=q' \quad\lor\quad (r\cdot q')+(w\cdot q')=(r+w)\cdot q'$ by

7: $(r\cdot(r'+w'))+(w\cdot(r'+w'))=(r+w)\cdot(r'+w')$

11: $\neg\, r'+w'=q' \quad\lor\quad \neg\,(r\cdot(r'+w'))+(w\cdot(r'+w'))=(r+w)\cdot(r'+w') \quad\lor\quad (r\cdot q')+(w\cdot q')=(r+w)\cdot q'$

22: $\neg\,(r\cdot q')+r'<(r+w)\cdot q'$ by

15: $r+w=q$

19: $\neg\, r+w=q \quad\lor\quad \neg\,(r\cdot q')+r'<(r+w)\cdot q'$

23: $r'<w\cdot q'$ by

16: $\neg\, 0=w$

18: $0=w \quad\lor\quad r'<w\cdot q'$

24: $(r\cdot r')+(r\cdot w')=r\cdot q'$ by

17: $r'+w'=q'$

20: $\neg\, r'+w'=q' \quad\lor\quad (r\cdot r')+(r\cdot w')=r\cdot q'$

25: $(r\cdot q')+(w\cdot q')=(r+w)\cdot q'$ by

17: $r' + w' = q'$

21: $\neg\, r' + w' = q' \quad \vee \quad (r \cdot q') + (w \cdot q') = (r + w) \cdot q'$

26: $\neg\, (r \cdot q') + (w \cdot q') = (r + w) \cdot q' \quad \vee \quad \neg\, (r \cdot q') + r' < (r \cdot q') + (w \cdot q')$ by

22: $\neg\, (r \cdot q') + r' < (r + w) \cdot q'$

13: $\neg\, (r \cdot q') + (w \cdot q') = (r + w) \cdot q' \quad \vee \quad \neg\, (r \cdot q') + r' < (r \cdot q') + (w \cdot q') \quad \vee \quad (r \cdot q') + r' < (r + w) \cdot q'$

27: $((r \cdot r') + (r \cdot w')) + r' < ((r \cdot r') + (r \cdot w')) + (w \cdot q')$ by

23: $r' < w \cdot q'$

9: $\neg\, r' < w \cdot q' \quad \vee \quad ((r \cdot r') + (r \cdot w')) + r' < ((r \cdot r') + (r \cdot w')) + (w \cdot q')$

28: $\neg\, ((r \cdot r') + (r \cdot w')) + r' < ((r \cdot r') + (r \cdot w')) + (w \cdot q') \quad \vee \quad (r \cdot q') + r' < (r \cdot q') + (w \cdot q')$ by

24: $(r \cdot r') + (r \cdot w') = r \cdot q'$

12: $\neg\, (r \cdot r') + (r \cdot w') = r \cdot q' \quad \vee \quad \neg\, ((r \cdot r') + (r \cdot w')) + r' < ((r \cdot r') + (r \cdot w')) + (w \cdot q') \quad \vee \quad (r \cdot q') + r' < (r \cdot q') + (w \cdot q')$

29: $\neg\, (r \cdot q') + r' < (r \cdot q') + (w \cdot q')$ by

25: $(r \cdot q') + (w \cdot q') = (r + w) \cdot q'$

26: $\neg\, (r \cdot q') + (w \cdot q') = (r + w) \cdot q' \quad \vee \quad \neg\, (r \cdot q') + r' < (r \cdot q') + (w \cdot q')$

30: $(r \cdot q') + r' < (r \cdot q') + (w \cdot q')$ by

27: $((r \cdot r') + (r \cdot w')) + r' < ((r \cdot r') + (r \cdot w')) + (w \cdot q')$

28: $\neg\, ((r \cdot r') + (r \cdot w')) + r' < ((r \cdot r') + (r \cdot w')) + (w \cdot q') \quad \vee \quad (r \cdot q') + r' < (r \cdot q') + (w \cdot q')$

31: QEA by

29: $\neg\, (r \cdot q') + r' < (r \cdot q') + (w \cdot q')$

30: $(r \cdot q') + r' < (r \cdot q') + (w \cdot q')$