Proof of Theorem 174

The theorem to be proved is

$x < y \quad \rightarrow \quad z + x < z + y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) < (y)] \quad \& \quad [\neg\, (z + x) < (z + y)]]$

Special cases of the hypothesis and previous results:

0: $x < y$ from H:x:y:z

1: $\neg\, z + x < z + y$ from H:x:y:z

2: $z + x = x + z$ from 98;z;x

3: $z + y = y + z$ from 98;z;y

4: $\neg\, x < y \quad \vee \quad x + z < y + z$ from 170;x;y;z

Equality substitutions:

5: $\neg\, z + x = x + z \quad \vee \quad z + x < z + y \quad \vee \quad \neg\, x + z < z + y$

6: $\neg\, z + y = y + z \quad \vee \quad x + z < z + y \quad \vee \quad \neg\, x + z < y + z$

Inferences:

7: $x + z < y + z$ by
0: $x < y$
4: $\neg\, x < y \quad \vee \quad x + z < y + z$

8: $\neg\, z + x = x + z \quad \vee \quad \neg\, x + z < z + y$ by
1: $\neg\, z + x < z + y$
5: $\neg\, z + x = x + z \quad \vee \quad z + x < z + y \quad \vee \quad \neg\, x + z < z + y$

9: $\neg\, x + z < z + y$ by
2: $z + x = x + z$
8: $\neg\, z + x = x + z \quad \vee \quad \neg\, x + z < z + y$

10: $x + z < z + y \quad \vee \quad \neg\, x + z < y + z$ by
3: $z + y = y + z$
6: $\neg\, z + y = y + z \quad \vee \quad x + z < z + y \quad \vee \quad \neg\, x + z < y + z$

11: $x + z < z + y$ by

7: $x + z < y + z$

10: $x + z < z + y \quad \vee \quad \neg\, x + z < y + z$

12: QEA by

9: $\neg\, x + z < z + y$

11: $x + z < z + y$