Proof of Theorem 173

The theorem to be proved is

$x \neq 0 \quad \& \quad y < z \quad \rightarrow \quad y < x \cdot z$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\,(x) = (0)] \quad \& \quad [(y) < (z)] \quad \& \quad [\neg\,(y) < (x \cdot z)]]$

Special cases of the hypothesis and previous results:

0: $\neg\, 0 = x$ from H:x:y:z

1: $y < z$ from H:x:y:z

2: $\neg\, y < x \cdot z$ from H:x:y:z

3: $0 = x \quad \vee \quad \mathrm{S}(\mathrm{P}x) = x$ from 22;x

4: $((\mathrm{P}x) \cdot z) + z = (\mathrm{S}(\mathrm{P}x)) \cdot z$ from 104;Px;z

5: $z + ((\mathrm{P}x) \cdot z) = ((\mathrm{P}x) \cdot z) + z$ from 98;$(\mathrm{P}x) \cdot z$;z

6: $z \leq z + ((\mathrm{P}x) \cdot z)$ from 71;z;$(\mathrm{P}x) \cdot z$

7: $\neg\, y < z \quad \vee \quad \neg\, z \leq x \cdot z \quad \vee \quad y < x \cdot z$ from 144;y;z;$x \cdot z$

Equality substitutions:

8: $\neg\, \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \neg\, ((\mathrm{P}x) \cdot z) + z = (\mathrm{S}(\mathrm{P}x)) \cdot z \quad \vee \quad ((\mathrm{P}x) \cdot z) + z = (x) \cdot z$

9: $\neg\, z + ((\mathrm{P}x) \cdot z) = ((\mathrm{P}x) \cdot z) + z \quad \vee \quad \neg\, z \leq z + ((\mathrm{P}x) \cdot z) \quad \vee \quad z \leq ((\mathrm{P}x) \cdot z) + z$

10: $\neg\, ((\mathrm{P}x) \cdot z) + z = x \cdot z \quad \vee \quad \neg\, z \leq ((\mathrm{P}x) \cdot z) + z \quad \vee \quad z \leq x \cdot z$

Inferences:

11: $\mathrm{S}(\mathrm{P}x) = x$ by

0: $\neg\, 0 = x$

3: $0 = x \quad \vee \quad \mathrm{S}(\mathrm{P}x) = x$

12: $\neg\, z \leq x \cdot z \quad \vee \quad y < x \cdot z$ by

1: $y < z$

7: $\neg\, y < z \quad \vee \quad \neg\, z \leq x \cdot z \quad \vee \quad y < x \cdot z$

13: $\neg\, z \leq x \cdot z$ by

2: $\neg\, y < x \cdot z$

12: $\neg\, z \leq x \cdot z \quad \vee \quad y < x \cdot z$

14: $\neg\, \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad ((\mathrm{P}x) \cdot z) + z = x \cdot z$ by

4: $((\mathrm{P}x) \cdot z) + z = (\mathrm{S}(\mathrm{P}x)) \cdot z$

8: $\neg\, \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \neg\, ((\mathrm{P}x) \cdot z) + z = (\mathrm{S}(\mathrm{P}x)) \cdot z \quad \vee \quad ((\mathrm{P}x) \cdot z) + z = x \cdot z$

15: $\neg\, z \leq z + ((\mathrm{P}x) \cdot z) \quad \vee \quad z \leq ((\mathrm{P}x) \cdot z) + z$ by

5: $z + ((\mathrm{P}x) \cdot z) = ((\mathrm{P}x) \cdot z) + z$

9: $\neg\, z + ((\mathrm{P}x) \cdot z) = ((\mathrm{P}x) \cdot z) + z \quad \vee \quad \neg\, z \leq z + ((\mathrm{P}x) \cdot z) \quad \vee \quad z \leq ((\mathrm{P}x) \cdot z) + z$

16: $z \leq ((\mathrm{P}x) \cdot z) + z$ by

6: $z \leq z + ((\mathrm{P}x) \cdot z)$

15: $\neg\, z \leq z + ((\mathrm{P}x) \cdot z) \quad \vee \quad z \leq ((\mathrm{P}x) \cdot z) + z$

17: $((\mathrm{P}x) \cdot z) + z = x \cdot z$ by

11: $\mathrm{S}(\mathrm{P}x) = x$

14: $\neg\, \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad ((\mathrm{P}x) \cdot z) + z = x \cdot z$

18: $\neg\, ((\mathrm{P}x) \cdot z) + z = x \cdot z \quad \vee \quad \neg\, z \leq ((\mathrm{P}x) \cdot z) + z$ by

13: $\neg\, z \leq x \cdot z$

10: $\neg\, ((\mathrm{P}x) \cdot z) + z = x \cdot z \quad \vee \quad \neg\, z \leq ((\mathrm{P}x) \cdot z) + z \quad \vee \quad z \leq x \cdot z$

19: $\neg\, ((\mathrm{P}x) \cdot z) + z = x \cdot z$ by

16: $z \leq ((\mathrm{P}x) \cdot z) + z$

18: $\neg\, ((\mathrm{P}x) \cdot z) + z = x \cdot z \quad \vee \quad \neg\, z \leq ((\mathrm{P}x) \cdot z) + z$

20: QEA by

17: $((\mathrm{P}x) \cdot z) + z = x \cdot z$

19: $\neg\, ((\mathrm{P}x) \cdot z) + z = x \cdot z$