Proof of Theorem 171

The theorem to be proved is

$$\mathrm{S}x = q + r \quad \& \quad q \text{ is a power of two} \quad \& \quad r < q \quad \rightarrow \quad q = \mathrm{Q}x \quad \& \quad r = \mathrm{R}x \qquad \star$$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(\mathrm{S}x) = (q + r)] \quad \& \quad [(q) \text{ is a power of two}] \quad \& \quad [(r) < (q)] \quad \& \quad [\neg\, (q) = (\mathrm{Q}x) \quad \vee \quad \neg\, (r) = (\mathrm{R}x)]]$

Special cases of the hypothesis and previous results:

0: $q + r = \mathrm{S}x$ from H:x:q:r

1: q is a power of two from H:x:q:r

2: $r < q$ from H:x:q:r

3: $\neg\, \mathrm{Q}x = q \quad \vee \quad \neg\, \mathrm{R}x = r$ from H:x:q:r

4: $q \leq q + r$ from 71;q;r

5: $2 \cdot q = q + q$ from 118;q

6: $r + q = q + r$ from 98;q;r

7: $\neg\, r < q \quad \vee \quad r + q < q + q$ from 170;r;q;q

8: $\neg\, q$ is a power of two $\vee \quad \neg\, q \leq \mathrm{S}x \quad \vee \quad \neg\, \mathrm{S}x < 2 \cdot q \quad \vee \quad \mathrm{Q}x = q$ from 159;q;x

9: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r \quad \vee \quad \mathrm{R}x = r$ from 120;Qx;r;Rx

10: $(\mathrm{Q}x) + (\mathrm{R}x) = \mathrm{S}x$ from 161;x

Equality substitutions:

11: $\neg\, q + r = \mathrm{S}x \quad \vee \quad \neg\, q \leq q + r \quad \vee \quad q \leq \mathrm{S}x$

12: $\neg\, q + r = \mathrm{S}x \quad \vee \quad \neg\, r + q = q + r \quad \vee \quad r + q = \mathrm{S}x$

13: $\neg\, \mathrm{Q}x = q \quad \vee \quad (\mathrm{Q}x) + r = \mathrm{S}x \quad \vee \quad \neg\, (q) + r = \mathrm{S}x$

14: $\neg\, 2 \cdot q = q + q \quad \vee \quad \mathrm{S}x < 2 \cdot q \quad \vee \quad \neg\, \mathrm{S}x < q + q$

15: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) = \mathrm{S}x \quad \vee \quad (\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r \quad \vee \quad \neg\, \mathrm{S}x = (\mathrm{Q}x) + r$

16: $\neg\, r + q = \mathrm{S}x \quad \vee \quad \neg\, r + q < q + q \quad \vee \quad \mathrm{S}x < q + q$

Inferences:

17: $\neg\, q \leq q + r \quad \vee \quad q \leq \mathrm{S}x$ by

0: $q + r = \mathrm{S}x$

11: $\neg\, q + r = \mathrm{S}x \quad \vee \quad \neg\, q \leq q + r \quad \vee \quad q \leq \mathrm{S}x$

18: $\neg\, r + q = q + r \quad \vee \quad r + q = \mathrm{S}x$ by

0: $q + r = \mathrm{S}x$

12: $\neg\, q + r = \mathrm{S}x \quad \vee \quad \neg\, r + q = q + r \quad \vee \quad r + q = \mathrm{S}x$

19: $\neg\, \mathrm{Q}x = q \quad \vee \quad (\mathrm{Q}x) + r = \mathrm{S}x$ by

0: $q + r = \mathrm{S}x$

13: $\neg\, \mathrm{Q}x = q \quad \vee \quad (\mathrm{Q}x) + r = \mathrm{S}x \quad \vee \quad \neg\, q + r = \mathrm{S}x$

20: $\neg\, q \leq \mathrm{S}x \quad \vee \quad \neg\, \mathrm{S}x < 2 \cdot q \quad \vee \quad \mathrm{Q}x = q$ by

1: q is a power of two

8: $\neg\, q$ is a power of two $\quad \vee \quad \neg\, q \leq \mathrm{S}x \quad \vee \quad \neg\, \mathrm{S}x < 2 \cdot q \quad \vee \quad \mathrm{Q}x = q$

21: $r + q < q + q$ by

2: $r < q$

7: $\neg\, r < q \quad \vee \quad r + q < q + q$

22: $q \leq \mathrm{S}x$ by

4: $q \leq q + r$

17: $\neg\, q \leq q + r \quad \vee \quad q \leq \mathrm{S}x$

23: $\mathrm{S}x < 2 \cdot q \quad \vee \quad \neg\, \mathrm{S}x < q + q$ by

5: $2 \cdot q = q + q$

14: $\neg\, 2 \cdot q = q + q \quad \vee \quad \mathrm{S}x < 2 \cdot q \quad \vee \quad \neg\, \mathrm{S}x < q + q$

24: $r + q = \mathrm{S}x$ by

6: $r + q = q + r$

18: $\neg\, r + q = q + r \quad \vee \quad r + q = \mathrm{S}x$

25: $(\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r \quad \vee \quad \neg\, (\mathrm{Q}x) + r = \mathrm{S}x$ by

10: $(\mathrm{Q}x) + (\mathrm{R}x) = \mathrm{S}x$

15: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) = \mathrm{S}x \quad \vee \quad (\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r \quad \vee \quad \neg\, (\mathrm{Q}x) + r = \mathrm{S}x$

26: $\neg\, r + q = \mathrm{S}x \quad \vee \quad \mathrm{S}x < q + q$ by

21: $r + q < q + q$

16: $\neg\, r + q = \mathrm{S}x \quad \vee \quad \neg\, r + q < q + q \quad \vee \quad \mathrm{S}x < q + q$

27: $\neg\, \mathrm{S}x < 2 \cdot q \quad \vee \quad \mathrm{Q}x = q$ by

22: $q \leq \mathrm{S}x$

20: $\neg\, q \leq \mathrm{S}x \quad \vee \quad \neg\, \mathrm{S}x < 2 \cdot q \quad \vee \quad \mathrm{Q}x = q$

28: $\mathrm{S}x < q + q$ by

24: $r + q = \mathrm{S}x$

26: $\neg\, r + q = \mathrm{S}x \quad \vee \quad \mathrm{S}x < q + q$

29: $\mathrm{S}x < 2 \cdot q$ by

28: $\mathrm{S}x < q + q$

23: $\mathrm{S}x < 2 \cdot q \quad \vee \quad \neg\, \mathrm{S}x < q + q$

30: $\mathrm{Q}x = q$ by

29: $\mathrm{S}x < 2 \cdot q$

27: $\neg\, \mathrm{S}x < 2 \cdot q \quad \vee \quad \mathrm{Q}x = q$

31: $\neg\, \mathrm{R}x = r$ by

30: $\mathrm{Q}x = q$

3: $\neg\, \mathrm{Q}x = q \quad \vee \quad \neg\, \mathrm{R}x = r$

32: $(\mathrm{Q}x) + r = \mathrm{S}x$ by

30: $\mathrm{Q}x = q$

19: $\neg\, \mathrm{Q}x = q \quad \vee \quad (\mathrm{Q}x) + r = \mathrm{S}x$

33: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r$ by

31: $\neg\, \mathrm{R}x = r$

9: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r \quad \vee \quad \mathrm{R}x = r$

34: $(\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r$ by

32: $(\mathrm{Q}x) + r = \mathrm{S}x$

25: $(\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r \quad \vee \quad \neg\, (\mathrm{Q}x) + r = \mathrm{S}x$

35: QEA by

33: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r$

34: $(\mathrm{Q}x) + (\mathrm{R}x) = (\mathrm{Q}x) + r$