

Proof of Theorem 169

The theorem to be proved is

$$x + y \leq x + z \rightarrow y \leq z$$

Suppose the theorem does not hold. Then, with the variables held fixed,

$$(H) \quad [(x + y) \leq (x + z)] \quad \& \quad [\neg (y) \leq (z)]$$

Special cases of the hypothesis and previous results:

- 0: $x + y \leq x + z$ from H: $x:y:z$
- 1: $\neg y \leq z$ from H: $x:y:z$
- 2: $\neg x + y \leq x + z \vee (x + y) + w = x + z$ from [167](#); $x + y;x + z:w$
- 3: $x + (y + w) = (x + y) + w$ from [72](#); $x;y:w$
- 4: $\neg x + (y + w) = x + z \vee y + w = z$ from [120](#); $x;y + w;z$
- 5: $y \leq y + w$ from [71](#); $y:w$

Equality substitutions:

- 6: $\neg (x + y) + w = x + z \vee \neg x + (y + w) = (x + y) + w \vee x + (y + w) = x + z$
- 7: $\neg y + w = z \vee \neg y \leq y + w \vee y \leq z$

Inferences:

- 8: $(x + y) + w = x + z$ by
 - 0: $x + y \leq x + z$
 - 2: $\neg x + y \leq x + z \vee (x + y) + w = x + z$
- 9: $\neg y + w = z \vee \neg y \leq y + w$ by
 - 1: $\neg y \leq z$
 - 7: $\neg y + w = z \vee \neg y \leq y + w \vee y \leq z$
- 10: $\neg (x + y) + w = x + z \vee x + (y + w) = x + z$ by
 - 3: $x + (y + w) = (x + y) + w$
 - 6: $\neg (x + y) + w = x + z \vee \neg x + (y + w) = (x + y) + w \vee x + (y + w) = x + z$

- 11: $\neg y + w = z$ by
5: $y \leq y + w$
9: $\neg y + w = z \vee \neg y \leq y + w$
- 12: $x + (y + w) = x + z$ by
8: $(x + y) + w = x + z$
10: $\neg (x + y) + w = x + z \vee x + (y + w) = x + z$
- 13: $\neg x + (y + w) = x + z$ by
11: $\neg y + w = z$
4: $\neg x + (y + w) = x + z \vee y + w = z$
- 14: *QEA* by
12: $x + (y + w) = x + z$
13: $\neg x + (y + w) = x + z$