Proof of Theorem 168

The theorem to be proved is

$x + y \leq x + z \quad \rightarrow \quad y \leq z$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x + y) \leq (x + z)] \quad \& \quad [\neg\, (y) \leq (z)]]$

Special cases of the hypothesis and previous results:

0: $x + y \leq x + z$ from H:x:y:z

1: $\neg\, y \leq z$ from H:x:y:z

2: $\neg\, x + y \leq x + z \quad \vee \quad (x + y) + w = x + z$ from 167;$x + y$;$x + z$:w

3: $x + (y + w) = (x + y) + w$ from 72;x;y;w

4: $\neg\, x + (y + w) = x + z \quad \vee \quad y + w = z$ from 120;x;$y + w$;z

5: $y \leq y + w$ from 71;y;w

Equality substitutions:

6: $\neg\, (x + y) + w = x + z \quad \vee \quad \neg\, x + (y + w) = (x + y) + w \quad \vee \quad x + (y + w) = x + z$

7: $\neg\, y + w = z \quad \vee \quad \neg\, y \leq y + w \quad \vee \quad y \leq z$

Inferences:

8: $(x + y) + w = x + z$ by
0: $x + y \leq x + z$
2: $\neg\, x + y \leq x + z \quad \vee \quad (x + y) + w = x + z$

9: $\neg\, y + w = z \quad \vee \quad \neg\, y \leq y + w$ by
1: $\neg\, y \leq z$
7: $\neg\, y + w = z \quad \vee \quad \neg\, y \leq y + w \quad \vee \quad y \leq z$

10: $\neg\, (x + y) + w = x + z \quad \vee \quad x + (y + w) = x + z$ by
3: $x + (y + w) = (x + y) + w$
6: $\neg\, (x + y) + w = x + z \quad \vee \quad \neg\, x + (y + w) = (x + y) + w \quad \vee \quad x + (y + w) = x + z$

11: $\neg\ y + w = z$ by

5: $y \leq y + w$

9: $\neg\ y + w = z \quad \vee \quad \neg\ y \leq y + w$

12: $x + (y + w) = x + z$ by

8: $(x + y) + w = x + z$

10: $\neg\ (x + y) + w = x + z \quad \vee \quad x + (y + w) = x + z$

13: $\neg\ x + (y + w) = x + z$ by

11: $\neg\ y + w = z$

4: $\neg\ x + (y + w) = x + z \quad \vee \quad y + w = z$

14: QEA by

12: $x + (y + w) = x + z$

13: $\neg\ x + (y + w) = x + z$