Proof of Theorem 167

The theorem to be proved is

$x \leq y \quad \rightarrow \quad \exists w{\leq}y[x + w = y]$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $\forall w[[(x) \leq (y)] \quad \& \quad [\neg\,(w) \leq (y) \quad \vee \quad \neg\,(x + w) = (y)]]$

Special cases of the hypothesis and previous results:

0: $x \leq y$ from H:x:y;$y - x$

1: $\neg\, y - x \leq y \quad \vee \quad \neg\, x + (y - x) = y$ from H:x:y;$y - x$

2: $\neg\, x \leq y \quad \vee \quad x + (y - x) = y$ from 68;x;y

3: $x + (y - x) = (y - x) + x$ from 98;x;$y - x$

4: $y - x \leq (y - x) + x$ from 71;$y - x$;x

Equality substitutions:

5: $\neg\, x + (y - x) = y \quad \vee \quad \neg\, x + (y - x) = (y - x) + x \quad \vee \quad y = (y - x) + x$

6: $\neg\,(y - x) + x = y \quad \vee \quad \neg\, y - x \leq (y - x) + x \quad \vee \quad y - x \leq y$

Inferences:

7: $x + (y - x) = y$ by
0: $x \leq y$
2: $\neg\, x \leq y \quad \vee \quad x + (y - x) = y$

8: $\neg\, x + (y - x) = y \quad \vee \quad (y - x) + x = y$ by
3: $x + (y - x) = (y - x) + x$
5: $\neg\, x + (y - x) = y \quad \vee \quad \neg\, x + (y - x) = (y - x) + x \quad \vee \quad (y - x) + x = y$

9: $\neg\,(y - x) + x = y \quad \vee \quad y - x \leq y$ by
4: $y - x \leq (y - x) + x$
6: $\neg\,(y - x) + x = y \quad \vee \quad \neg\, y - x \leq (y - x) + x \quad \vee \quad y - x \leq y$

10: $\neg\, y - x \leq y$ by
7: $x + (y - x) = y$
1: $\neg\, y - x \leq y \quad \vee \quad \neg\, x + (y - x) = y$

11: $(y - x) + x = y$ by

7: $x + (y - x) = y$

8: $\neg\, x + (y - x) = y \quad \vee \quad (y - x) + x = y$

12: $\neg\, (y - x) + x = y$ by

10: $\neg\, y - x \leq y$

9: $\neg\, (y - x) + x = y \quad \vee \quad y - x \leq y$

13: QEA by

11: $(y - x) + x = y$

12: $\neg\, (y - x) + x = y$