Proof of Theorem 165

The theorem to be proved is

$\mathrm{R}x < \mathrm{Q}x$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\, (\mathrm{R}x) < (\mathrm{Q}x)]]$

Special cases of the hypothesis and previous results:

0: $\neg\, \mathrm{R}x < \mathrm{Q}x$ from H:x

1: $(\mathrm{Q}x) + (\mathrm{R}x) = \mathrm{S}x$ from 161;x

2: $\mathrm{S}x < 2 \cdot (\mathrm{Q}x)$ from 158;x

3: $(\mathrm{Q}x) + (\mathrm{Q}x) = 2 \cdot (\mathrm{Q}x)$ from 118;Qx

4: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) < (\mathrm{Q}x) + (\mathrm{Q}x) \;\vee\; \mathrm{R}x < \mathrm{Q}x$ from 164;Qx;Rx;Qx

Equality substitutions:

5: $\neg\, (\mathrm{Q}x)+(\mathrm{Q}x) = 2\cdot(\mathrm{Q}x) \;\vee\; (\mathrm{Q}x)+(\mathrm{R}x) < (\mathrm{Q}x)+(\mathrm{Q}x) \;\vee\; \neg\, (\mathrm{Q}x)+(\mathrm{R}x) < 2\cdot(\mathrm{Q}x)$

6: $\neg\; \mathrm{S}x = (\mathrm{Q}x) + (\mathrm{R}x) \;\vee\; \neg\, (\mathrm{S}x) < 2 \cdot (\mathrm{Q}x) \;\vee\; \neg\, ((\mathrm{Q}x) + (\mathrm{R}x)) < 2 \cdot (\mathrm{Q}x)$

Inferences:

7: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) < (\mathrm{Q}x) + (\mathrm{Q}x)$ by
0: $\neg\, \mathrm{R}x < \mathrm{Q}x$
4: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) < (\mathrm{Q}x) + (\mathrm{Q}x) \;\vee\; \mathrm{R}x < \mathrm{Q}x$

8: $\neg\, \mathrm{S}x < 2 \cdot (\mathrm{Q}x) \;\vee\; (\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$ by
1: $(\mathrm{Q}x) + (\mathrm{R}x) = \mathrm{S}x$
6: $\neg\, (\mathrm{Q}x) + (\mathrm{R}x) = \mathrm{S}x \;\vee\; \neg\, \mathrm{S}x < 2 \cdot (\mathrm{Q}x) \;\vee\; (\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$

9: $(\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$ by
2: $\mathrm{S}x < 2 \cdot (\mathrm{Q}x)$
8: $\neg\, \mathrm{S}x < 2 \cdot (\mathrm{Q}x) \;\vee\; (\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$

10: $(\mathrm{Q}x) + (\mathrm{R}x) < (\mathrm{Q}x) + (\mathrm{Q}x) \;\vee\; \neg\, (\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$ by
3: $(\mathrm{Q}x) + (\mathrm{Q}x) = 2 \cdot (\mathrm{Q}x)$

5: $\neg\ (\mathrm{Q}x) + (\mathrm{Q}x) = 2 \cdot (\mathrm{Q}x) \quad \vee \quad (\mathrm{Q}x) + (\mathrm{R}x) < (\mathrm{Q}x) + (\mathrm{Q}x) \quad \vee \quad \neg\ (\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$

11: $\neg\ (\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$ by

7: $\neg\ (\mathrm{Q}x) + (\mathrm{R}x) < (\mathrm{Q}x) + (\mathrm{Q}x)$

10: $(\mathrm{Q}x) + (\mathrm{R}x) < (\mathrm{Q}x) + (\mathrm{Q}x) \quad \vee \quad \neg\ (\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$

12: QEA by

9: $(\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$

11: $\neg\ (\mathrm{Q}x) + (\mathrm{R}x) < 2 \cdot (\mathrm{Q}x)$