Proof of Theorem 15i

The theorem to be proved is

$$[x + y = 0 \quad \rightarrow \quad x = 0 \quad \& \quad y = 0] \quad \rightarrow \quad [x + \mathrm{S}y = 0 \quad \rightarrow \quad x = 0 \quad \& \quad \mathrm{S}y = 0]$$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\ (x + y) = (0) \quad \vee \quad (x) = (0)] \quad \& \quad [\neg\ (x + y) = (0) \quad \vee \quad (y) = (0)]$
$\& \quad [(x + (\mathrm{S}y)) = (0)] \quad \& \quad [\neg\ (x) = (0) \quad \vee \quad \neg\ (\mathrm{S}y) = (0)]]$

Special cases of the hypothesis and previous results:

0: $x + (\mathrm{S}y) = 0$ from H:x:y

1: $\mathrm{S}(x + y) = x + (\mathrm{S}y)$ from 12;x;y

2: $\neg\ \mathrm{S}(x + y) = 0$ from 3;$x + y$

Equality substitutions:

3: $\neg\ x + (\mathrm{S}y) = 0 \quad \vee \quad \neg\ \mathrm{S}(x + y) = x + (\mathrm{S}y) \quad \vee \quad \mathrm{S}(x + y) = 0$

Inferences:

4: $\neg\ \mathrm{S}(x + y) = x + (\mathrm{S}y) \quad \vee \quad \mathrm{S}(x + y) = 0$ by
0: $x + (\mathrm{S}y) = 0$
3: $\neg\ x + (\mathrm{S}y) = 0 \quad \vee \quad \neg\ \mathrm{S}(x + y) = x + (\mathrm{S}y) \quad \vee \quad \mathrm{S}(x + y) = 0$

5: $\mathrm{S}(x + y) = 0$ by
1: $\mathrm{S}(x + y) = x + (\mathrm{S}y)$
4: $\neg\ \mathrm{S}(x + y) = x + (\mathrm{S}y) \quad \vee \quad \mathrm{S}(x + y) = 0$

6: QEA by
2: $\neg\ \mathrm{S}(x + y) = 0$
5: $\mathrm{S}(x + y) = 0$