Proof of Theorem 156i

The theorem to be proved is

$\exists q{\le}\mathrm{S}x[\mathrm{p}_{150}(q,x)] \quad\rightarrow\quad \exists q_1{\le}\mathrm{SS}x[\mathrm{p}_{150}(q_1,\mathrm{S}x)]$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $\forall q_1[[(q) \le (\mathrm{S}x)] \quad \& \quad [\mathrm{p}_{150}((qx))] \quad \& \quad [\neg\,(q_1) \le (\mathrm{S}(\mathrm{S}x)) \quad \vee \quad \neg\,\mathrm{p}_{150}((q_1\mathrm{S}x))]]$

Special cases of the hypothesis and previous results:

0: $\mathrm{p}_{150}((qx))$ from H:x:q

1: $\neg\,\mathrm{p}_{150}((qx)) \quad \vee \quad q_1 \le \mathrm{S}(\mathrm{S}x)$ from 155;q;x:q_1

2: $\neg\,\mathrm{p}_{150}((qx)) \quad \vee \quad \mathrm{p}_{150}((q_1\mathrm{S}x))$ from 155;q;x:q_1

^{A} $\forall q_1[[(q) \le (\mathrm{S}x)] \quad \& \quad [\mathrm{p}_{150}((qx))] \quad \& \quad [\neg\,(q_1) \le (\mathrm{S}(\mathrm{S}x)) \quad \vee \quad \neg\,\mathrm{p}_{150}((q_1\mathrm{S}x))]]$ remnant from H:x:q

3: $\neg\,q_1 \le \mathrm{S}(\mathrm{S}x) \quad \vee \quad \neg\,\mathrm{p}_{150}((q_1\mathrm{S}x))$ from ^{A};q_1

Inferences:

4: $q_1 \le \mathrm{S}(\mathrm{S}x)$ by
- 0: $\mathrm{p}_{150}((qx))$
- 1: $\neg\,\mathrm{p}_{150}((qx)) \quad \vee \quad q_1 \le \mathrm{S}(\mathrm{S}x)$

5: $\mathrm{p}_{150}((q_1\mathrm{S}x))$ by
- 0: $\mathrm{p}_{150}((qx))$
- 2: $\neg\,\mathrm{p}_{150}((qx)) \quad \vee \quad \mathrm{p}_{150}((q_1\mathrm{S}x))$

6: $\neg\,\mathrm{p}_{150}((q_1\mathrm{S}x))$ by
- 4: $q_1 \le \mathrm{S}(\mathrm{S}x)$
- 3: $\neg\,q_1 \le \mathrm{S}(\mathrm{S}x) \quad \vee \quad \neg\,\mathrm{p}_{150}((q_1\mathrm{S}x))$

7: QEA by
- 5: $\mathrm{p}_{150}((q_1\mathrm{S}x))$
- 6: $\neg\,\mathrm{p}_{150}((q_1\mathrm{S}x))$