Proof of Theorem 149

The theorem to be proved is

$\neg\ [q \text{ is a power of two} \quad \& \quad q' \text{ is a power of two} \quad \& \quad q < q' \quad \& \quad q' < 2 \cdot q]$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $'[[(q) \text{ is a power of two}] \quad \& \quad [(q') \text{ is a power of two}] \quad \& \quad [(q) < (q')] \quad \& \quad [(q') < (2 \cdot q)]]$

Special cases of the hypothesis and previous results:

0: q is a power of two from H:q:q'

1: q' is a power of two from H:q:q'

2: $q < q'$ from H:q:q'

3: $q' < 2 \cdot q$ from H:q:q'

4: $\neg\ q$ is a power of two $\vee$ $2 \cdot q$ is a power of two from 135;q

5: $\neg\ q$ is a power of two $\vee$ $2 \uparrow x = q$ from $129^{\rightarrow}$;q:x

6: $\neg\ q'$ is a power of two $\vee$ $2 \uparrow y = q'$ from $129^{\rightarrow}$;q':y

7: $\neg\ 2 \cdot q$ is a power of two $\vee$ $2 \cdot q = 2 \uparrow z$ from $129^{\rightarrow}$;$2 \cdot q$:z

8: $2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x)$ from 126;2;x

9: $\neg\ 2 \uparrow x < 2 \uparrow y \quad \vee \quad x < y$ from 147;x;y

10: $\neg\ 2 \uparrow y < 2 \uparrow z \quad \vee \quad y < z$ from 147;y;z

11: $\neg\ 2 \uparrow (\mathrm{S}x) = 2 \uparrow z \quad \vee \quad \mathrm{S}x = z$ from 148;Sx;z

12: $\neg\ x < y \quad \vee \quad \neg\ y < \mathrm{S}x$ from 111;x;y

Equality substitutions:

13: $\neg\ 2 \uparrow x = q \quad \vee \quad \neg\ 2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x) \quad \vee \quad 2 \cdot (q) = 2 \uparrow (\mathrm{S}x)$

14: $\neg\ 2 \uparrow y = q' \quad \vee \quad 2 \uparrow x < 2 \uparrow y \quad \vee \quad \neg\ 2 \uparrow x < q'$

15: $\neg\ 2 \uparrow y = q' \quad \vee \quad 2 \uparrow y < 2 \uparrow z \quad \vee \quad \neg\ q' < 2 \uparrow z$

16: $\neg\ 2 \cdot q = 2 \uparrow z \quad \vee \quad \neg\ q' < 2 \cdot q \quad \vee \quad q' < 2 \uparrow z$

17: $\neg\ 2 \cdot q = 2 \uparrow z \quad \vee \quad \neg\ 2 \uparrow (\mathrm{S}x) = 2 \cdot q \quad \vee \quad 2 \uparrow (\mathrm{S}x) = 2 \uparrow z$

18: $\neg\, \mathrm{S}x = z \quad \vee \quad y < \mathrm{S}x \quad \vee \quad \neg\, y < z$

19: $\neg\ q = 2 \uparrow x \quad \vee \quad \neg\, (q) < q' \quad \vee \quad \neg\, (2 \uparrow x) < q'$

Inferences:

20: $2 \cdot q$ is a power of two by
0: q is a power of two
4: $\neg\, q$ is a power of two $\vee$ $2 \cdot q$ is a power of two

21: $2 \uparrow x = q$ by
0: q is a power of two
5: $\neg\, q$ is a power of two $\vee$ $2 \uparrow x = q$

22: $2 \uparrow y = q'$ by
1: q' is a power of two
6: $\neg\, q'$ is a power of two $\vee$ $2 \uparrow y = q'$

23: $\neg\, 2 \uparrow x = q \quad \vee \quad 2 \uparrow x < q'$ by
2: $q < q'$
19: $\neg\, 2 \uparrow x = q \quad \vee \quad \neg\, q < q' \quad \vee \quad 2 \uparrow x < q'$

24: $\neg\, 2 \cdot q = 2 \uparrow z \quad \vee \quad q' < 2 \uparrow z$ by
3: $q' < 2 \cdot q$
16: $\neg\, 2 \cdot q = 2 \uparrow z \quad \vee \quad \neg\, q' < 2 \cdot q \quad \vee \quad q' < 2 \uparrow z$

25: $\neg\, 2 \uparrow x = q \quad \vee \quad 2 \uparrow (\mathrm{S}x) = 2 \cdot q$ by
8: $2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x)$
13: $\neg\, 2 \uparrow x = q \quad \vee \quad \neg\, 2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x) \quad \vee \quad 2 \uparrow (\mathrm{S}x) = 2 \cdot q$

26: $2 \cdot q = 2 \uparrow z$ by
20: $2 \cdot q$ is a power of two
7: $\neg\, 2 \cdot q$ is a power of two $\vee$ $2 \cdot q = 2 \uparrow z$

27: $2 \uparrow x < q'$ by
21: $2 \uparrow x = q$
23: $\neg\, 2 \uparrow x = q \quad \vee \quad 2 \uparrow x < q'$

28: $2 \uparrow (\mathrm{S}x) = 2 \cdot q$ by
21: $2 \uparrow x = q$
25: $\neg\, 2 \uparrow x = q \quad \vee \quad 2 \uparrow (\mathrm{S}x) = 2 \cdot q$

29: $2 \uparrow x < 2 \uparrow y \quad \vee \quad \neg\, 2 \uparrow x < q'$ by

22: $2 \uparrow y = q'$

14: $\neg\, 2 \uparrow y = q' \quad \vee \quad 2 \uparrow x < 2 \uparrow y \quad \vee \quad \neg\, 2 \uparrow x < q'$

30: $2 \uparrow y < 2 \uparrow z \quad \vee \quad \neg\, q' < 2 \uparrow z$ by

22: $2 \uparrow y = q'$

15: $\neg\, 2 \uparrow y = q' \quad \vee \quad 2 \uparrow y < 2 \uparrow z \quad \vee \quad \neg\, q' < 2 \uparrow z$

31: $\neg\, 2 \uparrow (\mathrm{S}x) = 2 \cdot q \quad \vee \quad 2 \uparrow (\mathrm{S}x) = 2 \uparrow z$ by

26: $2 \cdot q = 2 \uparrow z$

17: $\neg\, 2 \cdot q = 2 \uparrow z \quad \vee \quad \neg\, 2 \uparrow (\mathrm{S}x) = 2 \cdot q \quad \vee \quad 2 \uparrow (\mathrm{S}x) = 2 \uparrow z$

32: $q' < 2 \uparrow z$ by

26: $2 \cdot q = 2 \uparrow z$

24: $\neg\, 2 \cdot q = 2 \uparrow z \quad \vee \quad q' < 2 \uparrow z$

33: $2 \uparrow x < 2 \uparrow y$ by

27: $2 \uparrow x < q'$

29: $2 \uparrow x < 2 \uparrow y \quad \vee \quad \neg\, 2 \uparrow x < q'$

34: $2 \uparrow (\mathrm{S}x) = 2 \uparrow z$ by

28: $2 \uparrow (\mathrm{S}x) = 2 \cdot q$

31: $\neg\, 2 \uparrow (\mathrm{S}x) = 2 \cdot q \quad \vee \quad 2 \uparrow (\mathrm{S}x) = 2 \uparrow z$

35: $2 \uparrow y < 2 \uparrow z$ by

32: $q' < 2 \uparrow z$

30: $2 \uparrow y < 2 \uparrow z \quad \vee \quad \neg\, q' < 2 \uparrow z$

36: $x < y$ by

33: $2 \uparrow x < 2 \uparrow y$

9: $\neg\, 2 \uparrow x < 2 \uparrow y \quad \vee \quad x < y$

37: $\mathrm{S}x = z$ by

34: $2 \uparrow (\mathrm{S}x) = 2 \uparrow z$

11: $\neg\, 2 \uparrow (\mathrm{S}x) = 2 \uparrow z \quad \vee \quad \mathrm{S}x = z$

38: $y < z$ by

35: $2 \uparrow y < 2 \uparrow z$

10: $\neg\, 2 \uparrow y < 2 \uparrow z \quad \vee \quad y < z$

39: $\neg\, y < \mathrm{S}x$ by

36: $x < y$

12: $\neg\, x < y \quad \vee \quad \neg\, y < \mathrm{S}x$

40: $y < \mathrm{S}x \quad \vee \quad \neg\, y < z$ by

37: $\mathrm{S}x = z$

18: $\neg\, \mathrm{S}x = z \quad \vee \quad y < \mathrm{S}x \quad \vee \quad \neg\, y < z$

41: $y < \mathrm{S}x$ by

38: $y < z$

40: $y < \mathrm{S}x \quad \vee \quad \neg\, y < z$

42: QEA by

39: $\neg\, y < \mathrm{S}x$

41: $y < \mathrm{S}x$