Proof of Theorem 147

The theorem to be proved is

$2 \uparrow x < 2 \uparrow y \quad \rightarrow \quad x < y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(2 \uparrow x) < (2 \uparrow y)] \quad \& \quad [\neg\, (x) < (y)]]$

Special cases of the hypothesis and previous results:

0: $2 \uparrow x < 2 \uparrow y$ from H:x:y

1: $\neg\, x < y$ from H:x:y

2: $x < y \quad \vee \quad y = x \quad \vee \quad y < x$ from 77;x;y

3: $\neg\, 2 \uparrow x < 2 \uparrow y \quad \vee \quad \neg\, 2 \uparrow y = 2 \uparrow x$ from $56^{\rightarrow}$;$2 \uparrow x$;$2 \uparrow y$

4: $\neg\, y < x \quad \vee \quad 2 \uparrow y < 2 \uparrow x$ from 145;y;x

5: $\neg\, 2 \uparrow x < 2 \uparrow y \quad \vee \quad \neg\, 2 \uparrow y \leq 2 \uparrow x$ from 80;$2 \uparrow x$;$2 \uparrow y$

6: $\neg\, 2 \uparrow y < 2 \uparrow x \quad \vee \quad 2 \uparrow y \leq 2 \uparrow x$ from $56^{\rightarrow}$;$2 \uparrow y$;$2 \uparrow x$

Equality substitutions:

7: $\neg\, y = x \quad \vee \quad 2 \uparrow y = 2 \uparrow x \quad \vee \quad \neg\, 2 \uparrow x = 2 \uparrow x$

Inferences:

8: $\neg\, 2 \uparrow y = 2 \uparrow x$ by
0: $2 \uparrow x < 2 \uparrow y$
3: $\neg\, 2 \uparrow x < 2 \uparrow y \quad \vee \quad \neg\, 2 \uparrow y = 2 \uparrow x$

9: $\neg\, 2 \uparrow y \leq 2 \uparrow x$ by
0: $2 \uparrow x < 2 \uparrow y$
5: $\neg\, 2 \uparrow x < 2 \uparrow y \quad \vee \quad \neg\, 2 \uparrow y \leq 2 \uparrow x$

10: $y = x \quad \vee \quad y < x$ by
1: $\neg\, x < y$
2: $x < y \quad \vee \quad y = x \quad \vee \quad y < x$

11: $\neg\, y = x$ by
8: $\neg\, 2 \uparrow y = 2 \uparrow x$
7: $\neg\, y = x \quad \vee \quad 2 \uparrow y = 2 \uparrow x$

12: $\neg\, 2 \uparrow y < 2 \uparrow x$ by

9: $\neg\, 2 \uparrow y \leq 2 \uparrow x$

6: $\neg\, 2 \uparrow y < 2 \uparrow x \quad \vee \quad 2 \uparrow y \leq 2 \uparrow x$

13: $y < x$ by

11: $\neg\, y = x$

10: $y = x \quad \vee \quad y < x$

14: $\neg\, y < x$ by

12: $\neg\, 2 \uparrow y < 2 \uparrow x$

4: $\neg\, y < x \quad \vee \quad 2 \uparrow y < 2 \uparrow x$

15: QEA by

13: $y < x$

14: $\neg\, y < x$