## Proof of Theorem 146

The theorem to be proved is

$x \leq y \quad \rightarrow \quad 2 \uparrow x \leq 2 \uparrow y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) \leq (y)] \quad \& \quad [\neg\, (2 \uparrow x) \leq (2 \uparrow y)]]$

### Special cases of the hypothesis and previous results:

0: $x \leq y$ from H:$x$:$y$

1: $\neg\, 2 \uparrow x \leq 2 \uparrow y$ from H:$x$:$y$

2: $\neg\, x \leq y \;\vee\; x < y \;\vee\; y = x$ from 61;$x$;$y$

3: $\neg\, x < y \;\vee\; 2 \uparrow x < 2 \uparrow y$ from 145;$x$;$y$

4: $\neg\, 2 \uparrow x < 2 \uparrow y \;\vee\; 2 \uparrow x \leq 2 \uparrow y$ from 56$^{\rightarrow}$;$2 \uparrow x$;$2 \uparrow y$

5: $2 \uparrow x \leq 2 \uparrow x$ from 60;$2 \uparrow x$

### Equality substitutions:

6: $\neg\, y = x \;\vee\; 2 \uparrow x \leq 2 \uparrow y \;\vee\; \neg\, 2 \uparrow x \leq 2 \uparrow x$

### Inferences:

7: $x < y \;\vee\; y = x$ by
0: $x \leq y$
2: $\neg\, x \leq y \;\vee\; x < y \;\vee\; y = x$

8: $\neg\, 2 \uparrow x < 2 \uparrow y$ by
1: $\neg\, 2 \uparrow x \leq 2 \uparrow y$
4: $\neg\, 2 \uparrow x < 2 \uparrow y \;\vee\; 2 \uparrow x \leq 2 \uparrow y$

9: $\neg\, y = x \;\vee\; \neg\, 2 \uparrow x \leq 2 \uparrow x$ by
1: $\neg\, 2 \uparrow x \leq 2 \uparrow y$
6: $\neg\, y = x \;\vee\; 2 \uparrow x \leq 2 \uparrow y \;\vee\; \neg\, 2 \uparrow x \leq 2 \uparrow x$

10: $\neg\, y = x$ by
5: $2 \uparrow x \leq 2 \uparrow x$
9: $\neg\, y = x \;\vee\; \neg\, 2 \uparrow x \leq 2 \uparrow x$

11: $\neg\, x < y$ by

8: $\neg\, 2 \uparrow x < 2 \uparrow y$

3: $\neg\, x < y \quad \vee \quad 2 \uparrow x < 2 \uparrow y$

12: $x < y$ by

10: $\neg\, y = x$

7: $x < y \quad \vee \quad y = x$

13: $QEA$ by

11: $\neg\, x < y$

12: $x < y$