Proof of Theorem 145

The theorem to be proved is

$x < y \quad \rightarrow \quad 2 \uparrow x < 2 \uparrow y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) < (y)] \quad \& \quad [\neg\, (2 \uparrow x) < (2 \uparrow y)]]$

Special cases of the hypothesis and previous results:

0: $x < y$ from H:x:y

1: $\neg\, 2 \uparrow x < 2 \uparrow y$ from H:x:y

2: $\neg\, x < y \quad \vee \quad x + (y - x) = y$ from 108;x;y

3: $\neg\, x < y \quad \vee \quad \neg\, y - x = 0$ from 108;x;y

4: $(2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow (x + (y - x))$ from 136;2;x;$y - x$

5: $y - x = 0 \quad \vee \quad 2 \leq 2 \uparrow (y - x)$ from 140;$y - x$

6: $\neg\, 2 \leq 2 \uparrow (y - x) \quad \vee \quad (2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x))$ from 142;2;$2 \uparrow (y - x)$;$2 \uparrow x$

7: $\neg\, 2 \uparrow x = 0$ from 133;x

8: $2 \uparrow x = 0 \quad \vee \quad 2 \uparrow x < (2 \uparrow x) \cdot 2$ from 143;$2 \uparrow x$

9: $\neg\, 2 \uparrow x < (2 \uparrow x) \cdot 2 \quad \vee \quad \neg\, (2 \uparrow x) \cdot 2 \leq 2 \uparrow y \quad \vee \quad 2 \uparrow x < 2 \uparrow y$ from 144;$2 \uparrow x$;$(2 \uparrow x) \cdot 2$;$2 \uparrow y$

Equality substitutions:

10: $\neg\, x+(y-x) = y \quad \vee \quad \neg\, (2\uparrow x)\cdot(2\uparrow(y-x)) = 2\uparrow(x+(y-x)) \quad \vee \quad (2\uparrow x)\cdot(2\uparrow(y-x)) = 2\uparrow(y)$

11: $\neg\, (2\uparrow x)\cdot(2\uparrow(y-x)) = 2\uparrow y \quad \vee \quad \neg\, (2\uparrow x)\cdot 2 \leq (2\uparrow x)\cdot(2\uparrow(y-x)) \quad \vee \quad (2\uparrow x)\cdot 2 \leq 2\uparrow y$

Inferences:

12: $x + (y - x) = y$ by

0: $x < y$

2: $\neg\, x < y \quad \vee \quad x + (y - x) = y$

13: $\neg\, y - x = 0$ by

0: $x < y$

3: $\neg\, x < y \quad \vee \quad \neg\, y - x = 0$

14: $\neg\, 2 \uparrow x < (2 \uparrow x) \cdot 2 \quad \vee \quad \neg\, (2 \uparrow x) \cdot 2 \leq 2 \uparrow y$ by

1: $\neg\, 2 \uparrow x < 2 \uparrow y$

9: $\neg\, 2 \uparrow x < (2 \uparrow x) \cdot 2 \quad \vee \quad \neg\, (2 \uparrow x) \cdot 2 \leq 2 \uparrow y \quad \vee \quad 2 \uparrow x < 2 \uparrow y$

15: $\neg\, x + (y - x) = y \quad \vee \quad (2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow y$ by

4: $(2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow (x + (y - x))$

10: $\neg\, x + (y - x) = y \quad \vee \quad \neg\, (2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow (x + (y - x)) \quad \vee \quad (2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow y$

16: $2 \uparrow x < (2 \uparrow x) \cdot 2$ by

7: $\neg\, 2 \uparrow x = 0$

8: $2 \uparrow x = 0 \quad \vee \quad 2 \uparrow x < (2 \uparrow x) \cdot 2$

17: $(2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow y$ by

12: $x + (y - x) = y$

15: $\neg\, x + (y - x) = y \quad \vee \quad (2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow y$

18: $2 \leq 2 \uparrow (y - x)$ by

13: $\neg\, y - x = 0$

5: $y - x = 0 \quad \vee \quad 2 \leq 2 \uparrow (y - x)$

19: $\neg\, (2 \uparrow x) \cdot 2 \leq 2 \uparrow y$ by

16: $2 \uparrow x < (2 \uparrow x) \cdot 2$

14: $\neg\, 2 \uparrow x < (2 \uparrow x) \cdot 2 \quad \vee \quad \neg\, (2 \uparrow x) \cdot 2 \leq 2 \uparrow y$

20: $\neg\, (2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x)) \quad \vee \quad (2 \uparrow x) \cdot 2 \leq 2 \uparrow y$ by

17: $(2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow y$

11: $\neg\, (2 \uparrow x) \cdot (2 \uparrow (y - x)) = 2 \uparrow y \quad \vee \quad \neg\, (2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x)) \quad \vee \quad (2 \uparrow x) \cdot 2 \leq 2 \uparrow y$

21: $(2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x))$ by

18: $2 \leq 2 \uparrow (y - x)$

6: $\neg\, 2 \leq 2 \uparrow (y - x) \quad \vee \quad (2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x))$

22: $\neg\, (2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x))$ by

19: $\neg\, (2 \uparrow x) \cdot 2 \leq 2 \uparrow y$

20: $\neg\, (2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x)) \quad \vee \quad (2 \uparrow x) \cdot 2 \leq 2 \uparrow y$

23: QEA by

21: $(2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x))$

22: $\neg\, (2 \uparrow x) \cdot 2 \leq (2 \uparrow x) \cdot (2 \uparrow (y - x))$