Proof of Theorem 144

The theorem to be proved is

$x < y \quad \& \quad y \leq z \quad \rightarrow \quad x < z$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) < (y)] \quad \& \quad [(y) \leq (z)] \quad \& \quad [\neg\, (x) < (z)]]$

Special cases of the hypothesis and previous results:

0: $x < y$ from H:x:y:z

1: $y \leq z$ from H:x:y:z

2: $\neg\, x < z$ from H:x:y:z

3: $\neg\, x < y \quad \vee \quad x \leq y$ from 56$^{\rightarrow}$;x;y

4: $\neg\, x \leq y \quad \vee \quad \neg\, y \leq z \quad \vee \quad x \leq z$ from 73;x;y;z

5: $x < z \quad \vee \quad \neg\, x \leq z \quad \vee \quad z = x$ from 56$^{\leftarrow}$;x;z

6: $\neg\, x < y \quad \vee \quad \neg\, y \leq x$ from 80;x;y

Equality substitutions:

7: $\neg\, z = x \quad \vee \quad \neg\, y \leq z \quad \vee \quad y \leq x$

Inferences:

8: $x \leq y$ by

0: $x < y$

3: $\neg\, x < y \quad \vee \quad x \leq y$

9: $\neg\, y \leq x$ by

0: $x < y$

6: $\neg\, x < y \quad \vee \quad \neg\, y \leq x$

10: $\neg\, x \leq y \quad \vee \quad x \leq z$ by

1: $y \leq z$

4: $\neg\, x \leq y \quad \vee \quad \neg\, y \leq z \quad \vee \quad x \leq z$

11: $\neg\, z = x \quad \vee \quad y \leq x$ by

1: $y \leq z$

7: $\neg\, z = x \quad \vee \quad \neg\, y \leq z \quad \vee \quad y \leq x$

12: $\neg\, x \leq z \quad \vee \quad z = x$ by

2: $\neg\, x < z$

5: $x < z \quad \vee \quad \neg\, x \leq z \quad \vee \quad z = x$

13: $x \leq z$ by

8: $x \leq y$

10: $\neg\, x \leq y \quad \vee \quad x \leq z$

14: $\neg\, z = x$ by

9: $\neg\, y \leq x$

11: $\neg\, z = x \quad \vee \quad y \leq x$

15: $z = x$ by

13: $x \leq z$

12: $\neg\, x \leq z \quad \vee \quad z = x$

16: QEA by

14: $\neg\, z = x$

15: $z = x$