Proof of Theorem 140

The theorem to be proved is

$x \neq 0 \quad \rightarrow \quad 2 \leq 2 \uparrow x$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\,(x) = (0)] \quad \& \quad [\neg\,(2) \leq (2 \uparrow x)]]$

Special cases of the hypothesis and previous results:

0: $\neg\, 0 = x$ from H:x

1: $\neg\, 2 \leq 2 \uparrow x$ from H:x

2: $S(S0) = 2$ from 116

3: $S0 = 1$ from 115

4: $x < 2 \uparrow x$ from 127;x

5: $\neg\, x < 2 \uparrow x \quad \vee \quad Sx \leq 2 \uparrow x$ from 114;x;$2 \uparrow x$

6: $0 = x \quad \vee \quad 1 \leq x$ from 139;x

7: $\neg\, 1 \leq x \quad \vee \quad S1 \leq Sx$ from 112;1;x

8: $\neg\, 2 \leq Sx \quad \vee \quad \neg\, Sx \leq 2 \uparrow x \quad \vee \quad 2 \leq 2 \uparrow x$ from 73;2;Sx;$2 \uparrow x$

Equality substitutions:

9: $\neg\, S0 = 1 \quad \vee \quad \neg\, S(S0) = 2 \quad \vee \quad S(1) = 2$

10: $\neg\, S1 = 2 \quad \vee \quad \neg\, S1 \leq Sx \quad \vee \quad 2 \leq Sx$

Inferences:

11: $1 \leq x$ by

0: $\neg\, 0 = x$

6: $0 = x \quad \vee \quad 1 \leq x$

12: $\neg\, 2 \leq Sx \quad \vee \quad \neg\, Sx \leq 2 \uparrow x$ by

1: $\neg\, 2 \leq 2 \uparrow x$

8: $\neg\, 2 \leq Sx \quad \vee \quad \neg\, Sx \leq 2 \uparrow x \quad \vee \quad 2 \leq 2 \uparrow x$

13: $\neg\, \mathrm{S}0 = 1 \quad \vee \quad \mathrm{S}1 = 2$ by

2: $\mathrm{S}(\mathrm{S}0) = 2$

9: $\neg\, \mathrm{S}0 = 1 \quad \vee \quad \neg\, \mathrm{S}(\mathrm{S}0) = 2 \quad \vee \quad \mathrm{S}1 = 2$

14: $\mathrm{S}1 = 2$ by

3: $\mathrm{S}0 = 1$

13: $\neg\, \mathrm{S}0 = 1 \quad \vee \quad \mathrm{S}1 = 2$

15: $\mathrm{S}x \leq 2 \uparrow x$ by

4: $x < 2 \uparrow x$

5: $\neg\, x < 2 \uparrow x \quad \vee \quad \mathrm{S}x \leq 2 \uparrow x$

16: $\mathrm{S}1 \leq \mathrm{S}x$ by

11: $1 \leq x$

7: $\neg\, 1 \leq x \quad \vee \quad \mathrm{S}1 \leq \mathrm{S}x$

17: $\neg\, \mathrm{S}1 \leq \mathrm{S}x \quad \vee \quad 2 \leq \mathrm{S}x$ by

14: $\mathrm{S}1 = 2$

10: $\neg\, \mathrm{S}1 = 2 \quad \vee \quad \neg\, \mathrm{S}1 \leq \mathrm{S}x \quad \vee \quad 2 \leq \mathrm{S}x$

18: $\neg\, 2 \leq \mathrm{S}x$ by

15: $\mathrm{S}x \leq 2 \uparrow x$

12: $\neg\, 2 \leq \mathrm{S}x \quad \vee \quad \neg\, \mathrm{S}x \leq 2 \uparrow x$

19: $2 \leq \mathrm{S}x$ by

16: $\mathrm{S}1 \leq \mathrm{S}x$

17: $\neg\, \mathrm{S}1 \leq \mathrm{S}x \quad \vee \quad 2 \leq \mathrm{S}x$

20: QEA by

18: $\neg\, 2 \leq \mathrm{S}x$

19: $2 \leq \mathrm{S}x$