Proof of Theorem 13i

The theorem to be proved is

$x + 0 = 0 + x \quad \rightarrow \quad \mathrm{S}x + 0 = 0 + \mathrm{S}x$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x + 0) = (0 + x)] \quad \& \quad [\neg\, ((\mathrm{S}x) + 0) = (0 + (\mathrm{S}x))]]$

Special cases of the hypothesis and previous results:

0: $0 + x = x + 0$ from H:x

1: $\neg\, (\mathrm{S}x) + 0 = 0 + (\mathrm{S}x)$ from H:x

2: $(\mathrm{S}x) + 0 = \mathrm{S}x$ from 12;Sx

3: $\mathrm{S}(0 + x) = 0 + (\mathrm{S}x)$ from 12;0;x

4: $x + 0 = x$ from 12;x

Equality substitutions:

5: $\neg\, 0 + x = x + 0 \quad \vee \quad \neg\, \mathrm{S}(0 + x) = 0 + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(x + 0) = 0 + (\mathrm{S}x)$

6: $\neg\, (\mathrm{S}x) + 0 = \mathrm{S}x \quad \vee \quad (\mathrm{S}x) + 0 = 0 + (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}x = 0 + (\mathrm{S}x)$

7: $\neg\, x + 0 = x \quad \vee \quad \neg\, \mathrm{S}(x + 0) = 0 + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(x) = 0 + (\mathrm{S}x)$

Inferences:

8: $\neg\, \mathrm{S}(0 + x) = 0 + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(x + 0) = 0 + (\mathrm{S}x)$ by
0: $0 + x = x + 0$
5: $\neg\, 0 + x = x + 0 \quad \vee \quad \neg\, \mathrm{S}(0 + x) = 0 + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(x + 0) = 0 + (\mathrm{S}x)$

9: $\neg\, (\mathrm{S}x) + 0 = \mathrm{S}x \quad \vee \quad \neg\, 0 + (\mathrm{S}x) = \mathrm{S}x$ by
1: $\neg\, (\mathrm{S}x) + 0 = 0 + (\mathrm{S}x)$
6: $\neg\, (\mathrm{S}x) + 0 = \mathrm{S}x \quad \vee \quad (\mathrm{S}x) + 0 = 0 + (\mathrm{S}x) \quad \vee \quad \neg\, 0 + (\mathrm{S}x) = \mathrm{S}x$

10: $\neg\, 0 + (\mathrm{S}x) = \mathrm{S}x$ by
2: $(\mathrm{S}x) + 0 = \mathrm{S}x$
9: $\neg\, (\mathrm{S}x) + 0 = \mathrm{S}x \quad \vee \quad \neg\, 0 + (\mathrm{S}x) = \mathrm{S}x$

11: $\mathrm{S}(x+0) = 0 + (\mathrm{S}x)$ by

3: $\mathrm{S}(0+x) = 0 + (\mathrm{S}x)$

8: $\neg\, \mathrm{S}(0+x) = 0 + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(x+0) = 0 + (\mathrm{S}x)$

12: $\neg\, \mathrm{S}(x+0) = 0 + (\mathrm{S}x) \quad \vee \quad 0 + (\mathrm{S}x) = \mathrm{S}x$ by

4: $x + 0 = x$

7: $\neg\, x + 0 = x \quad \vee \quad \neg\, \mathrm{S}(x+0) = 0 + (\mathrm{S}x) \quad \vee \quad 0 + (\mathrm{S}x) = \mathrm{S}x$

13: $\neg\, \mathrm{S}(x+0) = 0 + (\mathrm{S}x)$ by

10: $\neg\, 0 + (\mathrm{S}x) = \mathrm{S}x$

12: $\neg\, \mathrm{S}(x+0) = 0 + (\mathrm{S}x) \quad \vee \quad 0 + (\mathrm{S}x) = \mathrm{S}x$

14: QEA by

11: $\mathrm{S}(x+0) = 0 + (\mathrm{S}x)$

13: $\neg\, \mathrm{S}(x+0) = 0 + (\mathrm{S}x)$