Proof of Theorem 139

The theorem to be proved is

$x \neq 0 \quad \rightarrow \quad 1 \leq x$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\ (x) = (0)] \quad \& \quad [\neg\ (1) \leq (x)]]$

Special cases of the hypothesis and previous results:

0: $\neg\ 0 = x$ from H:x

1: $\neg\ 1 \leq x$ from H:x

2: $\mathrm{S}0 = 1$ from 115

3: $0 = x \quad \vee \quad \mathrm{S(P}x) = x$ from 22;x

4: $0 \leq \mathrm{P}x$ from 138;Px

5: $\neg\ 0 \leq \mathrm{P}x \quad \vee \quad \mathrm{S}0 \leq \mathrm{S(P}x)$ from 112;0;Px

Equality substitutions:

6: $\neg\ \mathrm{S(P}x) = x \quad \vee \quad \neg\ \mathrm{S}0 \leq \mathrm{S(P}x) \quad \vee \quad \mathrm{S}0 \leq x$

7: $\neg\ 1 = \mathrm{S}0 \quad \vee \quad (1) \leq x \quad \vee \quad \neg\ (\mathrm{S}0) \leq x$

Inferences:

8: $\mathrm{S(P}x) = x$ by
0: $\neg\ 0 = x$
3: $0 = x \quad \vee \quad \mathrm{S(P}x) = x$

9: $\neg\ \mathrm{S}0 = 1 \quad \vee \quad \neg\ \mathrm{S}0 \leq x$ by
1: $\neg\ 1 \leq x$
7: $\neg\ \mathrm{S}0 = 1 \quad \vee \quad 1 \leq x \quad \vee \quad \neg\ \mathrm{S}0 \leq x$

10: $\neg\ \mathrm{S}0 \leq x$ by
2: $\mathrm{S}0 = 1$
9: $\neg\ \mathrm{S}0 = 1 \quad \vee \quad \neg\ \mathrm{S}0 \leq x$

11: $\mathrm{S}0 \leq \mathrm{S}(\mathrm{P}x)$ by

4: $0 \leq \mathrm{P}x$

5: $\neg\, 0 \leq \mathrm{P}x \quad \vee \quad \mathrm{S}0 \leq \mathrm{S}(\mathrm{P}x)$

12: $\neg\, \mathrm{S}0 \leq \mathrm{S}(\mathrm{P}x) \quad \vee \quad \mathrm{S}0 \leq x$ by

8: $\mathrm{S}(\mathrm{P}x) = x$

6: $\neg\, \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \neg\, \mathrm{S}0 \leq \mathrm{S}(\mathrm{P}x) \quad \vee \quad \mathrm{S}0 \leq x$

13: $\neg\, \mathrm{S}0 \leq \mathrm{S}(\mathrm{P}x)$ by

10: $\neg\, \mathrm{S}0 \leq x$

12: $\neg\, \mathrm{S}0 \leq \mathrm{S}(\mathrm{P}x) \quad \vee \quad \mathrm{S}0 \leq x$

14: QEA by

11: $\mathrm{S}0 \leq \mathrm{S}(\mathrm{P}x)$

13: $\neg\, \mathrm{S}0 \leq \mathrm{S}(\mathrm{P}x)$