Proof of Theorem 136i

The theorem to be proved is

$x \uparrow (y+z) = (x \uparrow y) \cdot (x \uparrow z) \quad \to \quad x \uparrow (y + \mathrm{S}z) = (x \uparrow y) \cdot (x \uparrow \mathrm{S}z)$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x \uparrow (y+z)) = ((x \uparrow y) \cdot (x \uparrow z))] \quad \& \quad [\neg\, (x \uparrow (y + (\mathrm{S}z))) = ((x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)))]]$

Special cases of the hypothesis and previous results:

0: $(x \uparrow y) \cdot (x \uparrow z) = x \uparrow (y+z)$ from H:x:y:z

1: $\neg\, (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = x \uparrow (y + (\mathrm{S}z))$ from H:x:y:z

2: $\mathrm{S}(y+z) = y + (\mathrm{S}z)$ from 12;y;z

3: $x \cdot (x \uparrow (y+z)) = x \uparrow (\mathrm{S}(y+z))$ from 126;x;$y+z$

4: $x \cdot (x \uparrow z) = x \uparrow (\mathrm{S}z)$ from 126;x;z

5: $x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = (x \cdot (x \uparrow y)) \cdot (x \uparrow z)$ from 102;x;$x \uparrow y$;$x \uparrow z$

6: $(x \uparrow y) \cdot (x \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z)$ from 102;$x \uparrow y$;x;$x \uparrow z$

7: $x \cdot (x \uparrow y) = (x \uparrow y) \cdot x$ from 105;x;$x \uparrow y$

Equality substitutions:

8: $\neg\, (x \uparrow y) \cdot (x \uparrow z) = x \uparrow (y+z) \quad \vee \quad x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = x \uparrow (\mathrm{S}(y+z)) \quad \vee \quad \neg\, x \cdot (x \uparrow (y+z)) = x \uparrow (\mathrm{S}(y+z))$

9: $\neg\, \mathrm{S}(y+z) = y + (\mathrm{S}z) \quad \vee \quad \neg\, (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = x \uparrow (\mathrm{S}(y+z)) \quad \vee \quad (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = x \uparrow (y + (\mathrm{S}z))$

10: $\neg\, x \cdot (x \uparrow z) = x \uparrow (\mathrm{S}z) \quad \vee \quad \neg\, (x \uparrow y) \cdot (x \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z) \quad \vee \quad (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z)$

11: $\neg\, x \cdot (x \uparrow y) = (x \uparrow y) \cdot x \quad \vee \quad \neg\, x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = (x \cdot (x \uparrow y)) \cdot (x \uparrow z) \quad \vee \quad x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z)$

12: $\neg\, ((x \uparrow y) \cdot x) \cdot (x \uparrow z) = (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) \quad \vee \quad \neg\, x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z) \quad \vee \quad x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z))$

13: $\neg\, x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = x \uparrow (\mathrm{S}(y+z)) \quad \vee \quad \neg\, x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) \quad \vee \quad x \uparrow (\mathrm{S}(y+z)) = (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z))$

Inferences:

14: $x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = x \uparrow (\mathrm{S}(y + z)) \quad \vee \quad \neg\, x \cdot (x \uparrow (y + z)) = x \uparrow (\mathrm{S}(y + z))$ by

0: $(x \uparrow y) \cdot (x \uparrow z) = x \uparrow (y + z)$

8: $\neg\, (x \uparrow y) \cdot (x \uparrow z) = x \uparrow (y + z) \quad \vee \quad x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = x \uparrow (\mathrm{S}(y + z)) \quad \vee \quad \neg\, x \cdot (x \uparrow (y + z)) = x \uparrow (\mathrm{S}(y + z))$

15: $\neg\, \mathrm{S}(y + z) = y + (\mathrm{S}z) \quad \vee \quad \neg\, (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = x \uparrow (\mathrm{S}(y + z))$ by

1: $\neg\, (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = x \uparrow (y + (\mathrm{S}z))$

9: $\neg\, \mathrm{S}(y+z) = y+(\mathrm{S}z) \quad \vee \quad \neg\, (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z)) = x\uparrow(\mathrm{S}(y+z)) \quad \vee \quad (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z)) = x \uparrow (y + (\mathrm{S}z))$

16: $\neg\, (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = x \uparrow (\mathrm{S}(y + z))$ by

2: $\mathrm{S}(y + z) = y + (\mathrm{S}z)$

15: $\neg\, \mathrm{S}(y + z) = y + (\mathrm{S}z) \quad \vee \quad \neg\, (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = x \uparrow (\mathrm{S}(y + z))$

17: $x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = x \uparrow (\mathrm{S}(y + z))$ by

3: $x \cdot (x \uparrow (y + z)) = x \uparrow (\mathrm{S}(y + z))$

14: $x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = x \uparrow (\mathrm{S}(y + z)) \quad \vee \quad \neg\, x \cdot (x \uparrow (y + z)) = x \uparrow (\mathrm{S}(y + z))$

18: $\neg\, (x\uparrow y)\cdot(x\cdot(x\uparrow z)) = ((x\uparrow y)\cdot x)\cdot(x\uparrow z) \quad \vee \quad ((x\uparrow y)\cdot x)\cdot(x\uparrow z) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$ by

4: $x \cdot (x \uparrow z) = x \uparrow (\mathrm{S}z)$

10: $\neg\, x \cdot (x \uparrow z) = x \uparrow (\mathrm{S}z) \quad \vee \quad \neg\, (x \uparrow y) \cdot (x \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z) \quad \vee \quad ((x \uparrow y) \cdot x) \cdot (x \uparrow z) = (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z))$

19: $\neg\, x \cdot (x \uparrow y) = (x \uparrow y) \cdot x \quad \vee \quad x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z)$ by

5: $x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = (x \cdot (x \uparrow y)) \cdot (x \uparrow z)$

11: $\neg\, x \cdot (x \uparrow y) = (x \uparrow y) \cdot x \quad \vee \quad \neg\, x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = (x \cdot (x \uparrow y)) \cdot (x \uparrow z) \quad \vee \quad x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z)$

20: $((x \uparrow y) \cdot x) \cdot (x \uparrow z) = (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z))$ by

6: $(x \uparrow y) \cdot (x \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z)$

18: $\neg\, (x\uparrow y)\cdot(x\cdot(x\uparrow z)) = ((x\uparrow y)\cdot x)\cdot(x\uparrow z) \quad \vee \quad ((x\uparrow y)\cdot x)\cdot(x\uparrow z) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$

21: $x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z)$ by

7: $x \cdot (x \uparrow y) = (x \uparrow y) \cdot x$

19: $\neg\, x \cdot (x \uparrow y) = (x \uparrow y) \cdot x \quad \vee \quad x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = ((x \uparrow y) \cdot x) \cdot (x \uparrow z)$

22: $\neg\, x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = x \uparrow (\mathrm{S}(y + z)) \quad \vee \quad \neg\, x \cdot ((x \uparrow y) \cdot (x \uparrow z)) = (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z))$ by

16: $\neg\, (x \uparrow y) \cdot (x \uparrow (\mathrm{S}z)) = x \uparrow (\mathrm{S}(y + z))$

13: $\neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = x\uparrow(\mathrm{S}(y+z)) \quad\vee\quad \neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$
$\vee \quad (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z)) = x\uparrow(\mathrm{S}(y+z))$

23: $\neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$ by
17: $x\cdot((x\uparrow y)\cdot(x\uparrow z)) = x\uparrow(\mathrm{S}(y+z))$
22: $\neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = x\uparrow(\mathrm{S}(y+z)) \quad\vee\quad \neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$

24: $\neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = ((x\uparrow y)\cdot x)\cdot(x\uparrow z) \quad\vee\quad x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$
by
20: $((x\uparrow y)\cdot x)\cdot(x\uparrow z) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$
12: $\neg\, ((x\uparrow y)\cdot x)\cdot(x\uparrow z) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z)) \quad\vee\quad \neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = ((x\uparrow y)\cdot x)\cdot(x\uparrow z)$
$\vee \quad x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$

25: $x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$ by
21: $x\cdot((x\uparrow y)\cdot(x\uparrow z)) = ((x\uparrow y)\cdot x)\cdot(x\uparrow z)$
24: $\neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = ((x\uparrow y)\cdot x)\cdot(x\uparrow z) \quad\vee\quad x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$

26: QEA by
23: $\neg\, x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$
25: $x\cdot((x\uparrow y)\cdot(x\uparrow z)) = (x\uparrow y)\cdot(x\uparrow(\mathrm{S}z))$