Proof of Theorem 127i

The theorem to be proved is

$x < 2 \uparrow x \quad \rightarrow \quad \mathrm{S}x < 2 \uparrow \mathrm{S}x$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) < (2 \uparrow x)] \quad \& \quad [\neg\, (\mathrm{S}x) < (2 \uparrow (\mathrm{S}x))]]$

Special cases of the hypothesis and previous results:

0: $x < 2 \uparrow x$ from H:x

1: $\neg\, \mathrm{S}x < 2 \uparrow (\mathrm{S}x)$ from H:x

2: $2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x)$ from 126;2;x

3: $\neg\, x < 2 \uparrow x \quad \vee \quad \mathrm{S}x < 2 \cdot (2 \uparrow x)$ from 124;x;$2 \uparrow x$

Equality substitutions:

4: $\neg\, 2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}x < 2 \cdot (2 \uparrow x) \quad \vee \quad \mathrm{S}x < 2 \uparrow (\mathrm{S}x)$

Inferences:

5: $\mathrm{S}x < 2 \cdot (2 \uparrow x)$ by
  0: $x < 2 \uparrow x$
  3: $\neg\, x < 2 \uparrow x \quad \vee \quad \mathrm{S}x < 2 \cdot (2 \uparrow x)$

6: $\neg\, 2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}x < 2 \cdot (2 \uparrow x)$ by
  1: $\neg\, \mathrm{S}x < 2 \uparrow (\mathrm{S}x)$
  4: $\neg\, 2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}x < 2 \cdot (2 \uparrow x) \quad \vee \quad \mathrm{S}x < 2 \uparrow (\mathrm{S}x)$

7: $\neg\, \mathrm{S}x < 2 \cdot (2 \uparrow x)$ by
  2: $2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x)$
  6: $\neg\, 2 \cdot (2 \uparrow x) = 2 \uparrow (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}x < 2 \cdot (2 \uparrow x)$

8: QEA by
  5: $\mathrm{S}x < 2 \cdot (2 \uparrow x)$
  7: $\neg\, \mathrm{S}x < 2 \cdot (2 \uparrow x)$