Proof of Theorem 122

The theorem to be proved is

$x \leq y \quad \& \quad y < z \quad \rightarrow \quad x < z$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) \leq (y)] \quad \& \quad [(y) < (z)] \quad \& \quad [\neg\ (x) < (z)]]$

Special cases of the hypothesis and previous results:

0: $x \leq y$ from H:x:y:z

1: $y < z$ from H:x:y:z

2: $\neg\ x < z$ from H:x:y:z

3: $\neg\ y < z \quad \vee \quad y \leq z$ from 56$^{\rightarrow}$;y;z

4: $\neg\ x \leq y \quad \vee \quad \neg\ y \leq z \quad \vee \quad x \leq z$ from 73;x;y;z

5: $x < z \quad \vee \quad \neg\ x \leq z \quad \vee \quad z = x$ from 56$^{\leftarrow}$;x;z

6: $\neg\ y < x \quad \vee \quad \neg\ x \leq y$ from 80;y;x

Equality substitutions:

7: $\neg\ z = x \quad \vee \quad \neg\ y < z \quad \vee \quad y < x$

Inferences:

8: $\neg\ y \leq z \quad \vee \quad x \leq z$ by
0: $x \leq y$
4: $\neg\ x \leq y \quad \vee \quad \neg\ y \leq z \quad \vee \quad x \leq z$

9: $\neg\ y < x$ by
0: $x \leq y$
6: $\neg\ y < x \quad \vee \quad \neg\ x \leq y$

10: $y \leq z$ by
1: $y < z$
3: $\neg\ y < z \quad \vee \quad y \leq z$

11: $\neg\ z = x \quad \vee \quad y < x$ by
1: $y < z$
7: $\neg\ z = x \quad \vee \quad \neg\ y < z \quad \vee \quad y < x$

12: $\neg\, x \leq z \quad \vee \quad z = x$ by

2: $\neg\, x < z$

5: $x < z \quad \vee \quad \neg\, x \leq z \quad \vee \quad z = x$

13: $\neg\, z = x$ by

9: $\neg\, y < x$

11: $\neg\, z = x \quad \vee \quad y < x$

14: $x \leq z$ by

10: $y \leq z$

8: $\neg\, y \leq z \quad \vee \quad x \leq z$

15: $\neg\, x \leq z$ by

13: $\neg\, z = x$

12: $\neg\, x \leq z \quad \vee \quad z = x$

16: QEA by

14: $x \leq z$

15: $\neg\, x \leq z$