Proof of Theorem 121

The theorem to be proved is

$y \neq 0 \quad \rightarrow \quad x < x + y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\,(y) = (0)] \quad \& \quad [\neg\,(x) < (x + y)]]$

Special cases of the hypothesis and previous results:

0: $\neg\, 0 = y$ from H:y:x

1: $\neg\, x < x + y$ from H:y:x

2: $x \leq x + y$ from 71;x;y

3: $x < x + y \quad \vee \quad \neg\, x \leq x + y \quad \vee \quad x + y = x$ from $56^{\leftarrow}$;x;$x + y$

4: $\neg\, x + 0 = x + y \quad \vee \quad 0 = y$ from 120;x;0;y

5: $x + 0 = x$ from 12;x

Equality substitutions:

6: $\neg\, x + y = x \quad \vee \quad x + 0 = x + y \quad \vee \quad \neg\, x + 0 = x$

Inferences:

7: $\neg\, x + 0 = x + y$ by
0: $\neg\, 0 = y$
4: $\neg\, x + 0 = x + y \quad \vee \quad 0 = y$

8: $\neg\, x \leq x + y \quad \vee \quad x + y = x$ by
1: $\neg\, x < x + y$
3: $x < x + y \quad \vee \quad \neg\, x \leq x + y \quad \vee \quad x + y = x$

9: $x + y = x$ by
2: $x \leq x + y$
8: $\neg\, x \leq x + y \quad \vee \quad x + y = x$

10: $\neg\, x + y = x \quad \vee \quad x + 0 = x + y$ by
5: $x + 0 = x$
6: $\neg\, x + y = x \quad \vee \quad x + 0 = x + y \quad \vee \quad \neg\, x + 0 = x$

11: $\neg\, x + y = x$ by

7: $\neg\, x + 0 = x + y$

10: $\neg\, x + y = x \quad \vee \quad x + 0 = x + y$

12: QEA by

9: $x + y = x$

11: $\neg\, x + y = x$