Proof of Theorem 119i

The theorem to be proved is

$[y + x = z + x \to y = z] \quad \to \quad [y + \mathrm{S}x = z + \mathrm{S}x \to y = z]$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\, (y+x) = (z+x) \quad \vee \quad (y) = (z)] \quad \& \quad [(y+(\mathrm{S}x)) = (z+(\mathrm{S}x))] \quad \& \quad [\neg\, (y) = (z)]]$

Special cases of the hypothesis and previous results:

0: $\neg\, z + x = y + x \quad \vee \quad z = y$ from H:y:x:z

1: $z + (\mathrm{S}x) = y + (\mathrm{S}x)$ from H:y:x:z

2: $\neg\, z = y$ from H:y:x:z

3: $\mathrm{S}(y + x) = y + (\mathrm{S}x)$ from 12;y;x

4: $\mathrm{S}(z + x) = z + (\mathrm{S}x)$ from 12;z;x

5: $\neg\, \mathrm{S}(z + x) = \mathrm{S}(y + x) \quad \vee \quad z + x = y + x$ from 4;$y + x$;$z + x$

Equality substitutions:

6: $\neg\, z + (\mathrm{S}x) = y + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(y + x) = z + (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}(y + x) = y + (\mathrm{S}x)$

7: $\neg\, \mathrm{S}(z + x) = z + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(z + x) = \mathrm{S}(y + x) \quad \vee \quad \neg\, z + (\mathrm{S}x) = \mathrm{S}(y + x)$

Inferences:

8: $\mathrm{S}(y + x) = z + (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}(y + x) = y + (\mathrm{S}x)$ by
1: $z + (\mathrm{S}x) = y + (\mathrm{S}x)$
6: $\neg\, z + (\mathrm{S}x) = y + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(y + x) = z + (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}(y + x) = y + (\mathrm{S}x)$

9: $\neg\, z + x = y + x$ by
2: $\neg\, z = y$
0: $\neg\, z + x = y + x \quad \vee \quad z = y$

10: $\mathrm{S}(y + x) = z + (\mathrm{S}x)$ by
3: $\mathrm{S}(y + x) = y + (\mathrm{S}x)$
8: $\mathrm{S}(y + x) = z + (\mathrm{S}x) \quad \vee \quad \neg\, \mathrm{S}(y + x) = y + (\mathrm{S}x)$

11: $\mathrm{S}(z+x) = \mathrm{S}(y+x) \quad \vee \quad \neg\, \mathrm{S}(y+x) = z + (\mathrm{S}x)$ by

4: $\mathrm{S}(z+x) = z + (\mathrm{S}x)$

7: $\neg\, \mathrm{S}(z+x) = z + (\mathrm{S}x) \quad \vee \quad \mathrm{S}(z+x) = \mathrm{S}(y+x) \quad \vee \quad \neg\, \mathrm{S}(y+x) = z + (\mathrm{S}x)$

12: $\neg\, \mathrm{S}(z+x) = \mathrm{S}(y+x)$ by

9: $\neg\, z + x = y + x$

5: $\neg\, \mathrm{S}(z+x) = \mathrm{S}(y+x) \quad \vee \quad z + x = y + x$

13: $\mathrm{S}(z+x) = \mathrm{S}(y+x)$ by

10: $\mathrm{S}(y+x) = z + (\mathrm{S}x)$

11: $\mathrm{S}(z+x) = \mathrm{S}(y+x) \quad \vee \quad \neg\, \mathrm{S}(y+x) = z + (\mathrm{S}x)$

14: QEA by

12: $\neg\, \mathrm{S}(z+x) = \mathrm{S}(y+x)$

13: $\mathrm{S}(z+x) = \mathrm{S}(y+x)$