Proof of Theorem 114

The theorem to be proved is

$x < y \quad \rightarrow \quad \mathrm{S}x \leq y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) < (y)] \quad \& \quad [\neg\, (\mathrm{S}x) \leq (y)]]$

Special cases of the hypothesis and previous results:

0: $x < y$ from H:x:y

1: $\neg\, \mathrm{S}x \leq y$ from H:x:y

2: $\neg\, x < y \quad \vee \quad x + (y - x) = y$ from 108;x;y

3: $\neg\, x < y \quad \vee \quad \neg\, y - x = 0$ from 108;x;y

4: $y - x = 0 \quad \vee \quad \mathrm{S}(\mathrm{P}(y - x)) = y - x$ from 22;$y - x$

5: $(\mathrm{S}x) + (\mathrm{P}(y - x)) = x + (\mathrm{S}(\mathrm{P}(y - x)))$ from 14;x;$\mathrm{P}(y - x)$

6: $\mathrm{S}x \leq (\mathrm{S}x) + (\mathrm{P}(y - x))$ from 71;$\mathrm{S}x$;$\mathrm{P}(y - x)$

Equality substitutions:

7: $\neg\, (\mathrm{S}x) + (\mathrm{P}(y - x)) = x + (\mathrm{S}(\mathrm{P}(y - x))) \quad \vee \quad \neg\, \mathrm{S}x \leq (\mathrm{S}x) + (\mathrm{P}(y - x)) \quad \vee \quad \mathrm{S}x \leq x + (\mathrm{S}(\mathrm{P}(y - x)))$

8: $\neg\, y = x + (y - x) \quad \vee \quad \mathrm{S}x \leq (y) \quad \vee \quad \neg\, \mathrm{S}x \leq (x + (y - x))$

9: $\neg\, y - x = \mathrm{S}(\mathrm{P}(y - x)) \quad \vee \quad \mathrm{S}x \leq x + ((y - x)) \quad \vee \quad \neg\, \mathrm{S}x \leq x + ((\mathrm{S}(\mathrm{P}(y - x))))$

Inferences:

10: $x + (y - x) = y$ by

0: $x < y$

2: $\neg\, x < y \quad \vee \quad x + (y - x) = y$

11: $\neg\, y - x = 0$ by

0: $x < y$

3: $\neg\, x < y \quad \vee \quad \neg\, y - x = 0$

12: $\neg\, x + (y - x) = y \quad \lor \quad \neg\, \mathrm{S}x \leq x + (y - x)$ by

1: $\neg\, \mathrm{S}x \leq y$

8: $\neg\, x + (y - x) = y \quad \lor \quad \mathrm{S}x \leq y \quad \lor \quad \neg\, \mathrm{S}x \leq x + (y - x)$

13: $\neg\, \mathrm{S}x \leq (\mathrm{S}x) + (\mathrm{P}(y - x)) \quad \lor \quad \mathrm{S}x \leq x + (\mathrm{S}(\mathrm{P}(y - x)))$ by

5: $(\mathrm{S}x) + (\mathrm{P}(y - x)) = x + (\mathrm{S}(\mathrm{P}(y - x)))$

7: $\neg\, (\mathrm{S}x) + (\mathrm{P}(y - x)) = x + (\mathrm{S}(\mathrm{P}(y - x))) \quad \lor \quad \neg\, \mathrm{S}x \leq (\mathrm{S}x) + (\mathrm{P}(y - x))$
$\lor \quad \mathrm{S}x \leq x + (\mathrm{S}(\mathrm{P}(y - x)))$

14: $\mathrm{S}x \leq x + (\mathrm{S}(\mathrm{P}(y - x)))$ by

6: $\mathrm{S}x \leq (\mathrm{S}x) + (\mathrm{P}(y - x))$

13: $\neg\, \mathrm{S}x \leq (\mathrm{S}x) + (\mathrm{P}(y - x)) \quad \lor \quad \mathrm{S}x \leq x + (\mathrm{S}(\mathrm{P}(y - x)))$

15: $\neg\, \mathrm{S}x \leq x + (y - x)$ by

10: $x + (y - x) = y$

12: $\neg\, x + (y - x) = y \quad \lor \quad \neg\, \mathrm{S}x \leq x + (y - x)$

16: $\mathrm{S}(\mathrm{P}(y - x)) = y - x$ by

11: $\neg\, y - x = 0$

4: $y - x = 0 \quad \lor \quad \mathrm{S}(\mathrm{P}(y - x)) = y - x$

17: $\neg\, \mathrm{S}(\mathrm{P}(y - x)) = y - x \quad \lor \quad \mathrm{S}x \leq x + (y - x)$ by

14: $\mathrm{S}x \leq x + (\mathrm{S}(\mathrm{P}(y - x)))$

9: $\neg\, \mathrm{S}(\mathrm{P}(y - x)) = y - x \quad \lor \quad \mathrm{S}x \leq x + (y - x) \quad \lor \quad \neg\, \mathrm{S}x \leq x + (\mathrm{S}(\mathrm{P}(y - x)))$

18: $\neg\, \mathrm{S}(\mathrm{P}(y - x)) = y - x$ by

15: $\neg\, \mathrm{S}x \leq x + (y - x)$

17: $\neg\, \mathrm{S}(\mathrm{P}(y - x)) = y - x \quad \lor \quad \mathrm{S}x \leq x + (y - x)$

19: QEA by

16: $\mathrm{S}(\mathrm{P}(y - x)) = y - x$

18: $\neg\, \mathrm{S}(\mathrm{P}(y - x)) = y - x$