Proof of Theorem 113

The theorem to be proved is

$x < y \quad \to \quad \mathrm{S}x < \mathrm{S}y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) < (y)] \quad \& \quad [\neg\, (\mathrm{S}x) < (\mathrm{S}y)]]$

Special cases of the hypothesis and previous results:

0: $x < y$ from H:x:y

1: $\neg\, \mathrm{S}x < \mathrm{S}y$ from H:x:y

2: $\neg\, x \leq y \quad \vee \quad \mathrm{S}x \leq \mathrm{S}y$ from 112;x;y

3: $\neg\, x < y \quad \vee \quad x \leq y$ from $56^{\Rightarrow}$;x;y

4: $\neg\, x < y \quad \vee \quad \neg\, y = x$ from $56^{\Rightarrow}$;x;y

5: $\mathrm{S}x < \mathrm{S}y \quad \vee \quad \neg\, \mathrm{S}x \leq \mathrm{S}y \quad \vee \quad \mathrm{S}y = \mathrm{S}x$ from $56^{\Leftarrow}$;Sx;Sy

6: $\neg\, \mathrm{S}y = \mathrm{S}x \quad \vee \quad y = x$ from 4;x;y

Inferences:

7: $x \leq y$ by
0: $x < y$
3: $\neg\, x < y \quad \vee \quad x \leq y$

8: $\neg\, y = x$ by
0: $x < y$
4: $\neg\, x < y \quad \vee \quad \neg\, y = x$

9: $\neg\, \mathrm{S}x \leq \mathrm{S}y \quad \vee \quad \mathrm{S}y = \mathrm{S}x$ by
1: $\neg\, \mathrm{S}x < \mathrm{S}y$
5: $\mathrm{S}x < \mathrm{S}y \quad \vee \quad \neg\, \mathrm{S}x \leq \mathrm{S}y \quad \vee \quad \mathrm{S}y = \mathrm{S}x$

10: $\mathrm{S}x \leq \mathrm{S}y$ by
7: $x \leq y$
2: $\neg\, x \leq y \quad \vee \quad \mathrm{S}x \leq \mathrm{S}y$

11: $\neg\, \mathrm{S}y = \mathrm{S}x$ by
8: $\neg\, y = x$
6: $\neg\, \mathrm{S}y = \mathrm{S}x \quad \vee \quad y = x$

12: $\mathrm{S}y = \mathrm{S}x$ by

10: $\mathrm{S}x \leq \mathrm{S}y$

9: $\neg\ \mathrm{S}x \leq \mathrm{S}y \quad \vee \quad \mathrm{S}y = \mathrm{S}x$

13: QEA by

11: $\neg\ \mathrm{S}y = \mathrm{S}x$

12: $\mathrm{S}y = \mathrm{S}x$