Proof of Theorem 110

The theorem to be proved is

$x \leq \mathrm{S}y \quad \rightarrow \quad x \leq y \quad \vee \quad x = \mathrm{S}y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $\quad [[(x) \leq (\mathrm{S}y)] \quad \& \quad [\neg\,(x) \leq (y)] \quad \& \quad [\neg\,(x) = (\mathrm{S}y)]]$

Special cases of the hypothesis and previous results:

0: $x \leq \mathrm{S}y$ from H:x:y

1: $\neg\, x \leq y$ from H:x:y

2: $\neg\, \mathrm{S}y = x$ from H:x:y

3: $\neg\, x \leq \mathrm{S}y \quad \vee \quad x < \mathrm{S}y \quad \vee \quad \mathrm{S}y = x$ from 61;x;Sy

4: $\neg\, x < \mathrm{S}y \quad \vee \quad x \leq y$ from 109;x;y

Inferences:

5: $x < \mathrm{S}y \quad \vee \quad \mathrm{S}y = x$ by
0: $x \leq \mathrm{S}y$
3: $\neg\, x \leq \mathrm{S}y \quad \vee \quad x < \mathrm{S}y \quad \vee \quad \mathrm{S}y = x$

6: $\neg\, x < \mathrm{S}y$ by
1: $\neg\, x \leq y$
4: $\neg\, x < \mathrm{S}y \quad \vee \quad x \leq y$

7: $x < \mathrm{S}y$ by
2: $\neg\, \mathrm{S}y = x$
5: $x < \mathrm{S}y \quad \vee \quad \mathrm{S}y = x$

8: QEA by
6: $\neg\, x < \mathrm{S}y$
7: $x < \mathrm{S}y$