Proof of Theorem 103i

The theorem to be proved is

$0 \cdot x = 0 \quad \rightarrow \quad 0 \cdot \mathrm{S}x = 0$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(0 \cdot x) = (0)] \quad \& \quad [\neg\, (0 \cdot (\mathrm{S}x)) = (0)]]$

Special cases of the hypothesis and previous results:

0: $0 \cdot x = 0$ from H:x

1: $\neg\, 0 \cdot (\mathrm{S}x) = 0$ from H:x

2: $(0 \cdot x) + 0 = 0 \cdot (\mathrm{S}x)$ from 100;0;x

3: $0 + 0 = 0$ from 12;0

Equality substitutions:

4: $\neg\, 0 \cdot x = 0 \quad \vee \quad \neg\, (0 \cdot x) + 0 = 0 \cdot (\mathrm{S}x) \quad \vee \quad (0) + 0 = 0 \cdot (\mathrm{S}x)$

5: $\neg\, 0 + 0 = 0 \quad \vee \quad \neg\, 0 \cdot (\mathrm{S}x) = 0 + 0 \quad \vee \quad 0 \cdot (\mathrm{S}x) = 0$

Inferences:

6: $\neg\, (0 \cdot x) + 0 = 0 \cdot (\mathrm{S}x) \quad \vee \quad 0 \cdot (\mathrm{S}x) = 0 + 0$ by
0: $0 \cdot x = 0$
4: $\neg\, 0 \cdot x = 0 \quad \vee \quad \neg\, (0 \cdot x) + 0 = 0 \cdot (\mathrm{S}x) \quad \vee \quad 0 \cdot (\mathrm{S}x) = 0 + 0$

7: $\neg\, 0 + 0 = 0 \quad \vee \quad \neg\, 0 \cdot (\mathrm{S}x) = 0 + 0$ by
1: $\neg\, 0 \cdot (\mathrm{S}x) = 0$
5: $\neg\, 0 + 0 = 0 \quad \vee \quad \neg\, 0 \cdot (\mathrm{S}x) = 0 + 0 \quad \vee \quad 0 \cdot (\mathrm{S}x) = 0$

8: $0 \cdot (\mathrm{S}x) = 0 + 0$ by
2: $(0 \cdot x) + 0 = 0 \cdot (\mathrm{S}x)$
6: $\neg\, (0 \cdot x) + 0 = 0 \cdot (\mathrm{S}x) \quad \vee \quad 0 \cdot (\mathrm{S}x) = 0 + 0$

9: $\neg\, 0 \cdot (\mathrm{S}x) = 0 + 0$ by
3: $0 + 0 = 0$
7: $\neg\, 0 + 0 = 0 \quad \vee \quad \neg\, 0 \cdot (\mathrm{S}x) = 0 + 0$

10: QEA by
8: $0 \cdot (\mathrm{S}x) = 0 + 0$
9: $\neg\, 0 \cdot (\mathrm{S}x) = 0 + 0$