Warning Signs of a Possible Collapse of Contemporary Mathematics

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I rejoice that we live in a world of boundless, infinite possibilities, one in which with Blake we can see a world in a grain of sand and a heaven in a wildflower, hold infinity in the palm of your hand and eternity in an hour. I rejoice that the sacred scriptures of our faith portray a God who listens to prayer, who loves us and longs to lead us. I rejoice that my chosen line of work, mathematics, has enabled me to bring into being new things that did not exist before, and to greet with wonder and awe many amazing inventions of my fellow workers. I rejoice that daily we live immersed in infinity, that we have the freedom not only to make choices but at times to be the agent, by will or by grace, to sing to the Lord a new song.

Is infinity real? For example, are there infinitely many numbers? Yes indeed. When my granddaughter was a preschooler she asked for a problem to solve. I gave her two seventeen-digit numbers, chosen arbitrarily except that no carrying would be involved in finding the sum. When she summed the two numbers correctly she was overjoyed to hear that she had solved a mathematical problem that no one had ever solved before.

The celebration of infinity is the celebration of life, of newness, of becoming, of the wonder of possibilities that cannot be listed in a finished static rubric. The very etymology of the word infinite is "unfinished". As Aristotle observed, infinity is always potential and never actual or completed.

So what are we to make of the contrasting notion of a completed infinity? I confess at the outset to the strong emotions of loathing and feeling of oppression that the contemplation of an actual infinity arouses in me. It is the antithesis of life, of newness, of becoming—it is finished.

Consider a cosmology in which space is actually infinite. Assuming Euclidean geometry for simplicity of discussion, divide space into cubes of side 100^{100} light years, called "local regions", and call two local regions "widely separated" in case they are at least $100^{100^{100}}$ light years apart. If the world is deterministic, then it is a tale told by an idiot, full of sound and fury, signifying nothing. Otherwise, chance plays a rôle. There can

be no causal influence between widely separated local regions, and it is a simple and unavoidable result of probability theory that if we have an actual infinity of independent trials of an event, then if the event is possible at all it is certain to occur, and certain to occur infinitely often. Thus there will be infinitely many local regions with meetings like ours in San Marino where exactly the same words are spoken, and since it is *possible* for it to snow in August in San Marino, there will be infinitely many local regions with a blizzard at a meeting otherwise just like ours. A world in which space is actually infinite is a tale told by infinitely many idiots, full of sound and fury, signifying everything conceivable.

Turning from cosmology to theology, I want to emphasize that in this paragraph I am speaking personally. I have no wish to belittle thoughts that others may find helpful in attempting partially to understand the infinite mystery of God, or even to appear to do so. I shall just describe some thoughts that I personally find unhelpful. I am uneasy when abstract concepts, such as "omniscient" and "omnipotent", are posited as attributes of God. To my mind, these are formal linguistic constructions, unrelated to anything in our experience, with no clear meaning and prone to paradox. ("I can find a stone too heavy to lift, which is something God can't do.") I have the same reservations about the concept of actual infinity. To me these are all cold and debatable notions, and I find their application to God unscriptural and unhelpful to my understanding or worship.

It is widely believed that there is a clear and correct theory of actual infinity in mathematics. Certainly if there is not, then there cannot be a clear and correct use of actual infinity in cosmology or any other branch of science. I want to examine that belief. Let's turn to the concept of actual infinity in mathematics.

Actual infinity entered fully into mathematics with the work of Georg Cantor. My introduction to Cantor's theory, and a thrilling one it was, occurred at the age of sixteen when I picked up a copy of Bertrand Russell's *Introduction to Mathematical Philosophy* in Italian translation. A set is said to be of cardinality $\aleph_0$ in case it can be put in one-to-one correspondence with the natural numbers 0, 1, 2, ... To illustrate the axiom of choice, Russell invents the tale of the millionaire who bought $\aleph_0$ pairs of shoes and $\aleph_0$ pairs of socks. Then he has $\aleph_0$ shoes (as opposed to pairs of shoes), since we can make a one-to-one correspondence between the shoes and the natural numbers: mark with 0 the left shoe of the first pair, with 1 the right shoe of the first pair, with 2 the left shoe of the second pair, and so on. But how, Russell asks, can we do this with socks, where there is nothing to distinguish one sock in a pair from

the other? This requires choosing a sock in each pair, and the axiom of choice permits this. This fable convinced me that the axiom of choice is true. Well, we are all Platonists in our youth.

The most impressive feature of Cantor's theory is that he showed that there are different sizes of infinity, by his famous diagonal argument. But Russell applied this argument to establish his paradox: the set of all sets that are not elements of themselves both is and is not an element of itself. Actually, Russell's paradox was in response to Frege's work, not Cantor's. Frege gave a clear and precise account of his work, making it possible for Russell to show it was wrong, whereas Cantor's work was in parts so vague and imprecise that, as Pauli said of another theory, it was not even wrong.

This is the first warning sign of trouble in contemporary mathematics: the intuitive notion of infinite sets leads to a contradiction.

Important progress was made by Zermelo, who wrote axioms for set theory, one of which was unclear. His work was later extended by Fraenkel and now we have ZFC, Zermelo-Fraenkel set theory with the axiom of choice, which is commonly taken as the foundational theory for contemporary mathematics. But how do we know that ZFC is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism). I took an informal poll among some students of foundations and by and large the going odds on the consistency of ZFC were only 100 to 1, a far cry from the certainty popularly attributed to mathematical knowledge.

But I don't want to say more about set theory. To my mind, the trouble, and I believe there *is* trouble, lies deeper. Let's consider arithmetic. The natural numbers, which henceforth I'll just call numbers, begin with 0 and each is followed by its successor, obtained by adding 1. The notion of successor is more basic than that of addition, so it is customary to introduce the symbol S for successor. Thus the numbers are pictured as follows:

$$\bullet \qquad \bullet \qquad \bullet \qquad \bullet \qquad \cdots$$
$$0 \qquad S0 \qquad SS0 \qquad SSS0 \qquad \cdots$$

Figure 1 : The tale of ω

The tale of ω goes on forever, so to endow it with physical reality we would need a cosmology in which space is actually infinite.

The principal tool for proving theorems in arithmetic is induction, which may be stated as follows: if a property of numbers holds for 0, and if it holds for the successor of every number for which it holds, then it holds for all numbers. This appears to be innocuous. Suppose that a

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property satisfies the two hypotheses (called respectively the basis and the induction step) and we want to prove that the property holds for some number, say SSS0. We know it holds for 0 by the basis, so it holds for S0 by the induction step, and it holds for SS0 and then for SSS0 for the same reason, concluding the proof.

One problem is that the notion of "property" is vague. We could reify it as a collection of numbers, but this leads us back into set theory. The other way, and this is what today is meant by Peano Arithmetic (P), is to introduce a formal language and replace the vague notion of property by the precise, syntactical and concrete, notion of a formula of the language. Then the axioms of P are:

1. Not $Sx = 0$.
2. If $Sx = Sy$ then $x = y$.
3. $x + 0 = x$.
4. $x + Sy = S(x + y)$.
5. $x \cdot 0 = 0$.
6. $x \cdot Sy = (x \cdot y) + x$.
7. If $\varphi(0)$, and if for all x, $\varphi(x)$ implies $\varphi(Sx)$, then for all x, $\varphi(x)$.

In 7, the induction axioms, φ is any formula of the language of arithmetic. This formulation of P is still not satisfactory because of the use of "Not", "if ... then", "implies", and "for all". Let's rewrite the axioms using the usual symbolism of mathematical logic:

1. $\neg\, Sx = 0$.
2. $Sx = Sy \rightarrow x = y$.
3. $x + 0 = x$.
4. $x + Sy = S(x + y)$.
5. $x \cdot 0 = 0$.
6. $x \cdot Sy = (x \cdot y) + x$.
7. $\varphi(0) \mathrel{\&} \forall x\,[\,\varphi(x) \rightarrow \varphi(Sx)\,] \rightarrow \forall x\,[\,\varphi(x)\,]$.

This is not a wanton move, made to keep the sacred mysteries within the priesthood, but an absolutely vital step. In the crowning achievement of his extraordinary career, David Hilbert created proof theory, which for the first time enabled one to discuss mathematical theories with the same precision and clarity that previously had been reserved for mathematical objects, such as groups and topological spaces. The key idea was the formalization of logic, introducing formal symbols, such as $\neg$, $\rightarrow$, and $\forall$—which as a useful mnemonic may be read as "not", "implies", and "for all"—that are not assigned any meaning; they are simply combined according to certain explicit rules. This separation of syntax

from semantics enables one to treat mathematical reasoning itself by the methods of mathematics. Surprisingly, the way to a deep understanding of mathematical reasoning lay in stripping it of meaning. All future progress in mathematical logic—by Gödel, Gentzen, Cohen, and others—depended on this fundamental insight. Having emphasized this point, from now on I'll largely use ordinary language for greater readability, but it must be taken as a surrogate for formal logical symbolism.

Formulas and proofs are combinations of symbols formed according to certain explicit rules. Details can be found in any book that treats the predicate calculus. This formalization of proof makes precise the notion of rigorous argument actually used in mathematical practice, and when properly presented it is far closer to actual practice than mathematicians usually think possible.

Notice that no concept of actual infinity occurs in P, or even in ZFC for that matter. The infinite Figure 1 is no part of the theory. This is not surprising, since actual infinity plays no part in mathematical practice (because mathematics is a human activity carried out in the world of daily life). Mathematicians of widely divergent views on the foundations of mathematics can and do agree as to whether a purported proof is indeed a proof; it is just a matter of checking. But each deep open problem in mathematics poses a challenge to confront potential infinity: can one find, among the infinite possibilities of correct reasoning, a proof?

How do we know that P is a consistent theory, free from contradiction? That is, how do we know that we cannot prove both a formula and its negation? The usual answer is this: the actual infinity of all numbers, as in Figure 1, provides a model of the theory; the axioms 1–7 are true in the model and the syntactical rules of the theory are truth-preserving: if the premises of a rule of inference are true, so is the conclusion of the rule. Hence every theorem of the theory is true, and since a formula and its negation cannot both be true, the theory is consistent.

To study this argument we must examine the notion of truth in arithmetic. I'll illustrate it by discussing the twin primes conjecture, one of the famous open problems of mathematics. Two primes are called twin primes if they differ by 2; for example, 11 and 13 are twin primes. The twin primes conjecture is that there are infinitely many twin primes. This can be expressed by a formula in the language of arithmetic: "for all n there exists p with p greater than n such that p and $p + 2$ are primes". How can we tell whether this is true or false by looking at Figure 1? We may have found a certain number n and searched fruitlessly after that for a very long time, acquiring the suspicion that the conjecture is false, but to be certain we would need to go on forever, completing an

actually infinite search. Or contrariwise, we may keep on finding larger and larger twin primes and acquire the suspicion that the conjecture is true, but to be certain we would need to go on forever, completing an actually infinite search. This of course is not possible, and some students of foundations deny that, in the phrase beloved by many philosophers, there is a fact of the matter as to whether the conjecture is true or false. But, someone may say, God is omniscient, and in the divine mind there is a fact of the matter. I want to discuss this assertion, which I'll call divine omniscience in arithmetic, from two points of view.

A mathematician's reverie. Wouldn't it be great to have a god who would do actually infinite searches for me and tell me whether formulas of arithmetic are true or false? But lo! here is a little brass one that will fit nicely on my desk. I'll question it: "Is the twin primes conjecture true?" "Yes." Well, that's good to know, in its way. "Is there a proof in Peano Arithmetic?" "Yes." OK, but what I really want is to see a proof. "In the standard encoding, is the first bit of the shortest proof 0?" "Yes." "Is the second bit 0?" "No." Aha! Then I know the second bit is 1. [Continuing this extended game of twenty questions, the mathematician obtains a proof.] Hooray! I'll be famous! ... But mathematics used to be fun, and this is more tedious than the income tax. I'll just mop up the remaining millennium questions and with my six million dollars I'll abandon mathematics and take up beekeeping.

This conceit says something about mathematicians. We are not much concerned with truth; what interests us is proof. And apart from the external trappings of fame and fortune, the driving motivation for doing mathematics is to have fun. I don't feel that this fact requires apology; just don't let the funding agencies know.

But the story tells us nothing about truth in arithmetic. Let's look at divine omniscience in arithmetic again. Consider a different question, "Did Hansel and Gretel drop an even number of bread crumbs or an odd number?" It must be one or the other; there is no third possibility. It is impossible to find a proof one way or another from the facts we know. Nevertheless, someone might say, there is a fact of the matter in the divine mind. But the fallacy here is obvious. The bread crumbs are just a product of human imagination; the story was simply made up. You can see where I am heading.

The notion of the actual infinity of all numbers is a product of human imagination; the story is simply made up. The tale of ω even has the structure of the traditional fairy tale: "Once upon a time there was a number called 0. It had a successor, which in turn had a successor, and all the successors had successors happily ever after."

Some mathematicians, the fundamentalists, believe in the literal inerrancy of the tale, while others, the formalists, do not. When mathematicians are doing mathematics, as opposed to talking about mathematics, it makes no difference: the theorems and proofs of the ones are indistinguishable from those of the others.

Let us examine the fundamentalist belief in the existence of the completed infinity ω in the light of monotheistic faith. It is part of monotheistic faith, as I understand it, that everything in creation is contingent; I AM WHO I AM is not constrained by necessity. Are we to believe that ω is contingent, that the truths of arithmetic might have been different had it pleased God to make them so? Or are we to believe that ω is uncreated—existing in its infinite magnitude by necessity, as it was in the beginning, is now, and ever shall be? But these are unreal questions, like "can $\aleph_2$ angels dance on the head of a pin?"

Hilbert, spurred on by the incisive criticisms of classical mathematics by his antagonist Brouwer, realized that the the usual argument for the consistency of P is unsatisfactory. A semantic proof of consistency, by appeal to a model, simply replaces the question of the consistency of a simpler theory, such as P, by the question of the consistency of the far more complicated set theory in which the notion of a model can be expressed. He proposed to establish the consistency of classical mathematics, beginning with arithmetic, by concrete syntactical means. It is well known that his program was overly ambitious and that it was shown to be impossible of realization by Gödel. But I want to focus on another aspect of the Hilbert program here. Hilbert is revered and reviled as the founder of formalism, but his formalism was only a tactic in his struggle against Brouwer to preserve classical mathematics: in his deepest beliefs he was a Platonist (what I have called, as a polemical ploy, a fundamentalist). Witness his saying, "No one shall expel us from the Paradise that Cantor has created." Hilbert's mistake, which a radical formalist would not have made, was to pose the problem of *proving* by finitary means the consistency of arithmetic, rather than to pose the problem of *investigating* by finitary means whether arithmetic is consistent.

Gödel's second incompleteness theorem is that P cannot be proved consistent by means expressible in P, provided that P is consistent. This important proviso is often omitted. This theorem I take to be the second warning sign of trouble in contemporary mathematics. Its straightforward significance is this: perhaps P is inconsistent. But this is not how his profound result was received, due to the a priori conviction of just about everyone that P *must* be consistent.

How can one doubt the consistency of P? Because of the untamed

power of induction, which goes far beyond the original intuitive justification for it. Peano Arithmetic allows for the introduction of primitive recursive functions, as follows. To introduce a new primitive recursive function f, say of two variables, one first posits a value for $f(x, 0)$ in terms of previously defined functions and then posits a formula for $f(x, Sy)$ in terms of $f(x, y)$ and previously defined functions. The coherence of this scheme depends on the assumption that all of the values introduced in this way denote numbers that can be expressed as numerals, meaning terms of the form SSS...0 as in Figure 1. This can be proved in P using induction, but in doing so we extend the meaning of induction, originally justified by intuition on numerals, to new kinds of numbers—exponential numbers, superexponential numbers, and so forth—that are themselves created by induction. The reasoning that primitive recursion as a computational scheme always terminates—i.e., that primitive recursive functions are total—is circular, for the number of steps required for the computation to terminate can only be expressed in terms of these new primitive recursive functions themselves. This is illustrated by a fable. A student went to a teacher and a dialogue ensued:

S. Will you teach me Arithmetic?

T. Gladly, my boy. 0 is a numeral; if x is a numeral, so is Sx. Numerals are used to count things.

S. I understand.

T. *Addition* is introduced as follows:

$$x + 0 = x,$$
$$x + Sy = S(x + y).$$

Try some addition.

S. I have encountered no difficulty in reducing sums of numerals to numerals.

T. *Multiplication* is introduced as follows:

$$x \cdot 0 = 0,$$
$$x \cdot Sy = x + (x \cdot y).$$

S. I've tried a few examples and reduced the products to numerals. But what about

$$SS0 \cdot \ \ldots \ \cdot SS0$$

with y occurrences of SS0, where y is a long numeral?

T. It reduces to the numeral denoted by $SS0 \uparrow y$.

S. What is $\uparrow$?

T. It is *exponentiation*, introduced by

$$x \uparrow 0 = S0,$$

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$$x \uparrow \mathrm{S}y = x \cdot (x \uparrow y).$$

S. I've tried a few very small examples, but what about

$$\mathrm{SS0} \uparrow \ \cdots \ \uparrow \mathrm{SS0}$$

with y occurrences of SS0?[1]

T. It reduces to the numeral that $\mathrm{SS0} \Uparrow y$ denotes, where $\Uparrow$ is *superexponentiation*, introduced by

$$x \Uparrow 0 = \mathrm{S0},$$
$$x \Uparrow \mathrm{S}y = x \uparrow (x \Uparrow y).$$

S. I've looked at a couple of non-trivial examples but can't seem to make them work. I'll accept it on your authority. But what about

$$\mathrm{SS0} \Uparrow \ \cdots \ \Uparrow \mathrm{SS0}$$

with y occurrences of SS0?

T. Introduce *supersuperexponentiation* ...

S. Excuse me, teacher, but I am having trouble following you. I raised the question as to whether an arbitrarily long product of copies of SS0 can be reduced to a numeral. You responded in terms of $\uparrow$, and to the same question about $\uparrow$ in terms of $\Uparrow$, and so forth. You seem to be presenting us with a sky-hook. We want to hang a heavy weight in midair, so we suspend it from a hook, which is itself suspended from a hook, which in turn is suspended from a hook, and so on forever.

T. Very good, my boy! That is an excellent metaphor—a sky-hook is just what the Higher Arithmetic gives us. And it accommodates not only the primitive recursive functions we have been discussing but much more. Would you like to learn about the Ackermann function? General recursive functions?

S. Thank you ma'am. Another day, perhaps.

Finitism is usually regarded as the most conservative of all positions on the foundations of mathematics, but finitism accepts all primitive recursive functions as being total on the grounds that the computations involved are finite. But I have argued that they are simply postulated to be finite, by a circular reasoning. The untamed use of induction in P is justifiable only by appeal to ω as a completed infinity. Finitism is the last refuge of the Platonist.

It is one thing to criticize a position but something else to show that it is problematic, so now I turn to a closer examination of finitism. Let us extend P by adjoining a new symbol, which in accord with the decision to use ordinary language I write as "is a counting number". We adjoin two new axioms:

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[1]Infix symbols are to be associated from right to left.

8. 0 is a counting number.

9. If x is a counting number, then Sx is a counting number.

Call the extended theory P′. It is easy to see that if P is consistent then so is P′. This is because we have not defined the new notion of a counting number, and we could if we wished define "x is a counting number" to mean "$x = x$", so that all numbers would be counting numbers and (8) and (9) would hold trivially. But we don't do this; we leave it as an undefined notion.

It follows from (8) and (9) that 0, S0, SS0, and so forth, are counting numbers. In fact, our intuitive understanding of "number" is the same as that of "counting number". But we cannot prove that all numbers are counting numbers. One might be tempted to try to do so by induction, but the induction axioms of arithmetic were postulated for formulas φ of the specified language of arithmetic and "is a counting number" is not in that language. Moreover, there is a simple semantic argument using Gödel's completeness theorem, which he proved shortly before his more famous incompleteness theorems, showing that one cannot prove that all numbers are counting numbers, or even that if x and y are counting numbers then so is $x + y$.

But we can do something almost as good, called a relativization scheme. We define a new notion, that of *additionable number*, and show that not only are additionable numbers counting numbers but that the sum of two additionable numbers is again an additionable number.

10. Definition. x is an additionable number in case for all counting numbers y, the sum $y + x$ is a counting number.

Then we have:

11. Theorem. If x is an additionable number, then x is a counting number.

Proof. Let x be an additionable number. By (8), 0 is a counting number. By (10) applied to $y = 0$, $0 + x$ is a counting number, but $0 + x = x$.

12. Theorem. 0 is an additionable number.

Proof. Let y be a counting number. By (10), we need to show that $y + 0$ is a counting number. But $y + 0 = y$.

13. Theorem. If x is an additionable number, so is Sx.

Proof. Let x be an additionable number and let y be a counting number; by (10), we need to show that $y + Sx$ is a counting number.

But $y + Sx = S(y + x)$. By (10), $y + x$ is a counting number, so by (9), $S(y + x)$ is indeed a counting number.

14. Theorem. If x_1 and x_2 are additionable numbers, so is $x_1 + x_2$.

Proof. Let x_1 and x_2 be additionable numbers and let y be a counting number. By (10), we need to show that $y + (x_1 + x_2)$ is a counting number. But $y + (x_1 + x_2) = (y + x_1) + x_2$ (the associative law for addition), and $y + x_1$ is a counting number by (10), so $(y + x_1) + x_2$ is indeed a counting number, also by (10).

We can define an even stronger notion, *multiplicable number*, and show that multiplicable numbers are not only counting numbers but that the sum and product of two multiplicable numbers is again a multiplicable number.

15. Definition. x is a multiplicable number in case for all additionable numbers y, the product $y \cdot x$ is an additionable number.

By the same kind of elementary reasoning used above, one can prove the following theorems:

16. Theorem. If x is a multiplicable number, then x is an additionable number.
17. Theorem. If x is a multiplicable number, then x is a counting number.
18. Theorem. 0 is a multiplicable number.
19. Theorem. If x is a multiplicable number, so is Sx.
20. Theorem. If x_1 and x_2 are multiplicable numbers, so is $x_1 + x_2$.
21. Theorem. If x_1 and x_2 are multiplicable numbers, so is $x_1 \cdot x_2$.

The proof of the last theorem uses the associativity of multiplication. The significance of all this is that addition and multiplication are unproblematic. We have defined a new notion, that of a multiplicable number, that is stronger than the notion of counting number, and proved that multiplicable numbers not only have successors that are multiplicable numbers, and hence counting numbers, but that the same is true for sums and products of multiplicable numbers. For any specific numeral SSS...0 we can quickly prove that it is a multiplicable number.

But now we come to a halt. If we attempt to define "exponentiable number" in the same spirit, we are unable to prove that if x_1 and x_2 are exponentiable numbers then so is $x_1 \uparrow x_2$. There is a radical difference between addition and multiplication on the one hand and exponentiation, superexponentiation, and so forth, on the other hand. The obstacle is that exponentiation is not associative; for example, $(2 \uparrow 2) \uparrow 3 = 4 \uparrow 3 = 64$ whereas $2 \uparrow (2 \uparrow 3) = 2 \uparrow 8 = 256$. For

any specific numeral SSS...0 we can indeed prove that it is an exponentiable number, but we cannot prove that the world of exponentiable numbers is closed under exponentiation. And superexponentiation leads us entirely away from the world of counting numbers.

The belief that exponentiation, superexponentiation, and so forth, applied to numerals yield numerals is just that—a belief. Here we have the third, and most serious, warning sign of trouble in contemporary mathematics.

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