Idempotency*

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Axiom 0.1 (Commutativity) $\forall(x_1, x_2)\ x_2 \cup x_1 = x_1 \cup x_2$.

Axiom 0.2 (Associativity) $\forall(x_1, x_2, x_3)\ (x_1 \cup x_2) \cup x_3 = x_1 \cup x_2 \cup x_3$.

Axiom 0.3 (Huntington's axiom) $\forall(x_1, x_2)\ \overline{x_1 \cup \overline{x_2}} \cup \overline{\overline{x_1} \cup \overline{x_2}} = x_2$.

Theorem 0.1 $a \cup a = a$.

Proof[1]. We show directly that

$$a \cup a = a. \tag{1}$$

Because of Huntington's axiom and by associativity

$$\forall(x_1, x_2, x_3)\ \overline{x_1 \cup \overline{x_2}} \cup \overline{\overline{x_1} \cup \overline{x_2}} \cup x_3 = x_2 \cup x_3. \tag{2}$$

We show directly that

$$\forall(x_1, x_2, x_3)\ x_2 \cup x_1 \cup x_3 = x_1 \cup x_2 \cup x_3. \tag{3}$$

Let us consider arbitrary v_1, v_2, v_3. Because of commutativity and by associativity $v_1 \cup v_2 \cup v_3 = (v_2 \cup v_1) \cup v_3 = v_2 \cup v_1 \cup v_3$. This completes the proof of (3).

Hence by Huntington's axiom

$$\forall(x_1, x_2, x_3)\ \overline{x_1 \cup \overline{x_3}} \cup x_2 \cup \overline{\overline{x_1} \cup \overline{x_3}} = x_2 \cup x_3. \tag{4}$$

Because of commutativity and by Huntington's axiom

$$\forall(x_1, x_2)\ \overline{\overline{x_2} \cup x_1} \cup \overline{\overline{x_1} \cup \overline{x_2}} = x_2. \tag{5}$$

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[1] Otter and Ilf

Hence by commutativity

$$\forall(x_1, x_2)\ \overline{\overline{x_2} \cup x_1} \cup \overline{\overline{x_2} \cup \overline{x_1}} = x_2. \tag{6}$$

Hence by (4)

$$\forall(x_1, x_2)\ \overline{x_1 \cup \overline{x_2}} \cup x_1 = \overline{\overline{x_1} \cup x_2} \cup x_2. \tag{7}$$

Hence by commutativity $\forall(x_1, x_2)\ \overline{\overline{x_2} \cup x_1} \cup x_1 = \overline{\overline{x_1} \cup x_2} \cup x_2$. Hence by commutativity $\forall(x_1, x_2)\ x_1 \cup \overline{\overline{x_2} \cup x_1} = \overline{\overline{x_1} \cup x_2} \cup x_2$. Hence by (2)

$$\forall(x_1, x_2)\ \overline{x_1 \cup \overline{x_2}} \cup x_1 \cup \overline{\overline{\overline{x_2} \cup x_1}} = x_2 \cup \overline{x_2}. \tag{8}$$

By (6) and by commutativity

$$\forall(x_1, x_2)\ \overline{\overline{x_2} \cup \overline{x_1}} \cup \overline{\overline{x_2} \cup x_1} = x_2. \tag{9}$$

Hence by (4) $\forall(x_1, x_2)\ \overline{\overline{x_2} \cup \overline{\overline{x_1}}} \cup x_1 = \overline{x_2 \cup \overline{x_1}} \cup x_2$. Hence by (2)

$$\forall(x_1, x_2)\ \overline{x_1 \cup \overline{\overline{x_2}}} \cup \overline{x_1 \cup \overline{x_2}} \cup x_1 = \overline{x_2} \cup x_2. \tag{10}$$

Because of commutativity and by Huntington's axiom

$$\forall(x_1, x_2)\ \overline{\overline{x_1} \cup \overline{x_2}} \cup \overline{\overline{x_2} \cup \overline{x_1}} = x_2. \tag{11}$$

Hence by (4)

$$\forall(x_1, x_2)\ \overline{x_1 \cup \overline{\overline{x_2}}} \cup x_1 = \overline{x_2 \cup \overline{\overline{x_1}}} \cup x_2. \tag{12}$$

Hence by (7) $\forall(x_1, x_2)\ \overline{\overline{x_1} \cup x_2} \cup x_2 = \overline{x_2 \cup \overline{x_1}} \cup x_2$. Hence by commutativity

$$\forall(x_1, x_2)\ x_2 \cup \overline{\overline{x_1} \cup x_2} = \overline{x_2 \cup \overline{x_1}} \cup x_2. \tag{13}$$

Because of commutativity and by associativity

$$\forall(x_1, x_2, x_3)\ x_3 \cup x_1 \cup x_2 = x_1 \cup x_2 \cup x_3. \tag{14}$$

Hence by (11)

$$\forall(x_1, x_2, x_3)\ \overline{\overline{x_3} \cup \overline{x_1}} \cup x_2 \cup \overline{x_1 \cup \overline{x_3}} = x_2 \cup x_3. \tag{15}$$

We show directly that

$$\forall(x_1, x_2)\ \overline{x_1} \cup x_1 = \overline{x_2} \cup x_2. \tag{16}$$

Let us consider arbitrary v_1, v_2. By (13), (15), and by (10) $\overline{v_2} \cup v_2 = \overline{\overline{v_2} \cup \overline{\overline{v_1}}} \cup \overline{\overline{v_2} \cup \overline{v_1}} \cup \overline{v_2} = \overline{v_1} \cup v_1$. This completes the proof of (16).

Hence by (11)

$$\forall(x_1, x_2)\ \overline{x_2 \cup \overline{\overline{x_2}} \cup \overline{\overline{x_1} \cup x_1}} = \overline{x_2}. \tag{17}$$

By (11) and by commutativity

$$\forall(x_1, x_2)\ \overline{\overline{x_2} \cup \overline{x_1}} \cup \overline{x_1 \cup \overline{x_2}} = x_2. \tag{18}$$

We show directly that

$$\forall\ x_1\ \overline{\overline{x_1}} = x_1. \tag{19}$$

Let us consider arbitrary v_1, v_2. By (16), (18), and by (17) $v_2 = \overline{\overline{v_2} \cup \overline{\overline{v_2}}} \cup \overline{\overline{v_1} \cup v_1} = \overline{\overline{v_2}}$. This completes the proof of (19).

Hence by (8)

$$\forall(x_1, x_2)\ \overline{\overline{x_1} \cup \overline{x_2}} \cup x_1 \cup \overline{x_2 \cup x_1} = x_2 \cup \overline{x_2}. \tag{20}$$

We show directly that

$$\forall(x_1, x_2, x_3)\ \overline{\overline{x_3} \cup \overline{x_1}} \cup x_2 \cup \overline{x_1 \cup x_3} = x_2 \cup \overline{x_3}. \tag{21}$$

Let us consider arbitrary v_1, v_2, v_3. By (19) and by (15) $v_3 \cup \overline{v_1} = \overline{\overline{v_1} \cup \overline{v_2}} \cup v_3 \cup \overline{v_2 \cup v_1} = \overline{v_1 \cup \overline{v_2}} \cup v_3 \cup \overline{v_2 \cup v_1}$. This completes the proof of (21).

Hence by (20)

$$\forall(x_1, x_2)\ x_1 \cup \overline{x_1} = x_2 \cup \overline{x_2}. \tag{22}$$

By (16) and by (2)

$$\forall(x_1, x_2, x_3)\ \overline{x_2 \cup \overline{x_3}} \cup \overline{x_1} \cup x_1 = x_3 \cup \overline{x_2} \cup \overline{x_3}. \tag{23}$$

We show directly that

$$\forall(x_1, x_2, x_3)\ x_1 \cup x_3 \cup x_2 = x_1 \cup x_2 \cup x_3. \tag{24}$$

Let us consider arbitrary v_1, v_2, v_3. By (3), by commutativity, and by associativity $v_2 \cup v_1 \cup v_3 = (v_2 \cup v_3) \cup v_1 = v_2 \cup v_3 \cup v_1$. This completes the proof of (24).

Hence by (16)

$$\forall(x_1, x_2, x_3)\ \overline{x_2 \cup x_1} \cup x_1 \cup x_2 = \overline{x_3} \cup x_3. \tag{25}$$

We show directly that

$$\forall(x_1, x_2)\ x_1 \cup \overline{x_1} \cup \overline{x_1} = \overline{x_2} \cup x_2. \tag{26}$$

Let us consider arbitrary v_1, v_2, v_3. By (22), (25), and by (23) $\overline{\overline{v_3}} \cup v_3 = \overline{v_1 \cup \overline{\overline{v_1}} \cup \overline{\overline{v_2}} \cup v_2} = v_1 \cup \overline{\overline{v_1}} \cup \overline{\overline{v_1}}$. This completes the proof of (26).

We show directly that

$$\forall(x_1, x_2)\ \overline{\overline{x_1}} \cup x_1 \cup x_1 = \overline{\overline{x_2}} \cup x_2. \tag{27}$$

Let us consider arbitrary v_1, v_2. By (19) and by (26) $\overline{\overline{v_2}} \cup v_2 = \overline{\overline{v_1}} \cup v_1 \cup \overline{\overline{\overline{v_1}}} = \overline{\overline{v_1}} \cup v_1 \cup v_1$. This completes the proof of (27).

By (14) and by commutativity

$$\forall(x_1, x_2, x_3)\ x_3 \cup x_2 \cup x_1 = x_1 \cup x_2 \cup x_3. \tag{28}$$

Therefore $\forall(x_1, x_2)\ x_1 \cup x_1 \cup \overline{\overline{x_1}} = \overline{\overline{x_2}} \cup x_2$. Hence by (22) $\forall(x_1, x_2, x_3)\ x_1 \cup x_2 \cup \overline{\overline{x_2}} = \overline{\overline{x_3}} \cup x_3$. Hence by (28)

$$\forall(x_1, x_2, x_3)\ \overline{\overline{x_1}} \cup x_1 \cup x_2 = \overline{\overline{x_3}} \cup x_3. \tag{29}$$

We show directly that

$$\forall(x_1, x_2)\ \overline{\overline{\overline{x_2}} \cup \overline{\overline{x_1}} \cup \overline{x_1 \cup \overline{\overline{x_2}}}} \cup \overline{\overline{x_2}} = x_1 \cup \overline{\overline{x_2}}. \tag{30}$$

Let us consider arbitrary v_1, v_2. By (11) and by associativity $v_1 \cup \overline{\overline{v_2}} = \overline{(\overline{\overline{v_2}} \cup \overline{\overline{v_1}}) \cup \overline{v_1 \cup \overline{\overline{v_2}}} \cup \overline{\overline{v_2}}} = \overline{\overline{v_2}} \cup \overline{\overline{v_1}} \cup \overline{v_1 \cup \overline{\overline{v_2}}} \cup \overline{\overline{v_2}}$. This completes the proof of (30).

We show directly that

$$\forall(x_1, x_2)\ \overline{\overline{\overline{x_1} \cup x_1}} \cup x_2 = x_2 \cup x_2. \tag{31}$$

Let us consider arbitrary v_1, v_2. By (29) and by (30) $\overline{\overline{\overline{v_1} \cup v_1}} \cup \overline{\overline{v_2}} = v_2 \cup \overline{\overline{v_2}}$. Hence by (19) $\overline{\overline{\overline{v_1} \cup v_1}} \cup v_2 = v_2 \cup \overline{\overline{v_2}} = v_2 \cup v_2$. This completes the proof of (31).

By (5) and by (4) $\forall(x_1, x_2)\ \overline{x_1 \cup \overline{\overline{x_2}}} \cup x_2 = \overline{\overline{x_2} \cup x_1} \cup x_2$. Hence by commutativity

$$\forall(x_1, x_2)\ x_2 \cup \overline{x_1 \cup \overline{\overline{x_2}}} = \overline{\overline{x_2} \cup x_1} \cup x_2. \tag{32}$$

We show directly that

$$\forall(x_1, x_2)\ \overline{\overline{\overline{x_1} \cup x_1}} \cup x_2 = x_2 \cup \overline{x_2 \cup x_2 \cup \overline{\overline{x_2}}}. \tag{33}$$

Let us consider arbitrary v_1, v_2. By (26), (32), (19), and by associativity

$$\begin{aligned}
\overline{\overline{\overline{v_1} \cup v_1}} \cup v_2 &= v_2 \cup \overline{(\overline{\overline{v_2}} \cup \overline{\overline{v_2}}) \cup \overline{v_2}} \\
&= v_2 \cup \overline{(v_2 \cup \overline{\overline{v_2}}) \cup \overline{v_2}} \\
&= v_2 \cup \overline{(v_2 \cup v_2) \cup \overline{v_2}} \\
\overline{\overline{\overline{v_1} \cup v_1}} \cup v_2 &= v_2 \cup \overline{v_2 \cup v_2 \cup \overline{\overline{v_2}}}.
\end{aligned}$$

This completes the proof of (33).

By (19) and by (29)

$$\forall(x_1, x_2, x_3)\ x_1 \cup \overline{x_1} = \overline{x_2} \cup x_2 \cup x_3. \tag{34}$$

We show directly that

$$\forall(x_1, x_2)\ \overline{\overline{x_1 \cup \overline{x_1}} \cup x_2} = x_2 \cup x_2. \tag{35}$$

Let us consider arbitrary v_1, v_2. By (34) and by (30) $\overline{v_1 \cup \overline{v_1}} \cup \overline{\overline{v_2}} = v_2 \cup \overline{\overline{v_2}}$. Hence by (19) $\overline{v_1 \cup \overline{v_1}} \cup v_2 = v_2 \cup \overline{\overline{v_2}} = v_2 \cup v_2$. This completes the proof of (35).

We show directly that

$$\forall(x_1, x_2)\ \overline{\overline{x_1} \cup x_1} \cup \overline{x_2 \cup x_2} = \overline{x_2}. \tag{36}$$

Let us consider arbitrary v_1, v_2. By (16), (9), and by (19) $\overline{v_2} = \overline{\overline{v_1} \cup v_1} \cup \overline{\overline{\overline{v_2}} \cup v_2} = \overline{\overline{v_1} \cup v_1} \cup \overline{v_2 \cup v_2}$. This completes the proof of (36).

Hence by (19)

$$\forall(x_1, x_2)\ \overline{x_1 \cup \overline{\overline{x_1}}} \cup \overline{x_2 \cup x_2} = \overline{x_2}. \tag{37}$$

By (12) and by commutativity $\forall(x_1, x_2)\ x_1 \cup \overline{x_1 \cup \overline{x_2}} = \overline{x_2 \cup \overline{x_1}} \cup x_2$. Hence by commutativity

$$\forall(x_1, x_2)\ x_2 \cup \overline{x_2 \cup \overline{x_1}} = x_1 \cup \overline{x_1 \cup \overline{x_2}}. \tag{38}$$

We show directly that

$$\forall(x_1, x_2)\ x_2 \cup \overline{x_2 \cup x_1 \cup \overline{x_1}} = x_2. \tag{39}$$

Let us consider arbitrary v_1, v_2. By (35), (38), and by (37) $v_2 \cup \overline{v_2 \cup \overline{\overline{v_1} \cup \overline{v_1}}} = \overline{v_1 \cup \overline{v_1}} \cup \overline{\overline{v_2} \cup \overline{v_2}} = \overline{\overline{v_2}}$. Hence by (19) $v_2 = v_2 \cup v_2 \cup \overline{\overline{v_1} \cup \overline{v_1}} = v_2 \cup \overline{v_2 \cup v_1 \cup \overline{v_1}}$. This completes the proof of (39).

Hence by (33) $\forall(x_1, x_2)\ \overline{\overline{x_1} \cup x_1} \cup x_2 = x_2$. Hence by (31) $\forall\ x_1\ x_1 \cup x_1 = x_1$. Thus we have completed the proof of (1).

q.e.d.