Distributivity*

Nelson

February 7, 1997

Axiom 0.1 (Commutativity) $\forall(x_1, x_2)\ x_2 \cup x_1 = x_1 \cup x_2$.

Axiom 0.2 (Associativity) $\forall(x_1, x_2, x_3)\ (x_1 \cup x_2) \cup x_3 = x_1 \cup x_2 \cup x_3$.

Axiom 0.3 (Huntington's axiom) $\forall(x_1, x_2)\ \overline{x_1 \cup \overline{x_2}} \cup \overline{\overline{x_1} \cup \overline{x_2}} = x_2$.

Axiom 0.4 (Idempotency) $\forall\ x_1\ x_1 \cup x_1 = x_1$.

Axiom 0.5 (Involution) $\forall\ x_1\ \overline{\overline{x_1}} = x_1$.

Theorem 0.1 $\overline{\overline{a} \cup \overline{b \cup c}} = \overline{\overline{a} \cup \overline{b}} \cup \overline{\overline{a} \cup \overline{c}}$.

Proof[1].

We show indirectly that

$$\overline{\overline{a} \cup \overline{b \cup c}} = \overline{\overline{a} \cup \overline{b}} \cup \overline{\overline{a} \cup \overline{c}}. \tag{1}$$

Let us assume that

$$\neg\ \overline{\overline{a} \cup \overline{b \cup c}} = \overline{\overline{a} \cup \overline{b}} \cup \overline{\overline{a} \cup \overline{c}}. \tag{2}$$

Hence by commutativity

$$\neg\ \overline{\overline{a} \cup \overline{c \cup b}} = \overline{\overline{a} \cup \overline{b}} \cup \overline{\overline{a} \cup \overline{c}}. \tag{3}$$

We show directly that

$$\forall(x_1, x_2)\ \overline{\overline{x_1 \cup x_2} \cup \overline{\overline{x_1} \cup x_2}} = \overline{x_2}. \tag{4}$$

Let us consider arbitrary v_1, v_2. Because of involution and by Huntington's axiom $\overline{v_1} = \overline{v_2 \cup v_1} \cup \overline{\overline{v_2} \cup \overline{v_1}} = \overline{v_2 \cup v_1} \cup \overline{\overline{v_2} \cup v_1}$. This completes the proof of (4).

*This manuscript was generated by ILF. The development of ILF was supported by the Deutsche Forschungsgemeinschaft. For information on ILF contact gehne@mathematik.hu-berlin.de.

[1] Otter and Ilf

We show directly that

$$\forall(x_1, x_2)\ \overline{x_1 \cup x_2 \cup \overline{\overline{x_1} \cup x_2}} \cup x_2 = \overline{x_1} \cup x_2. \tag{5}$$

Let us consider arbitrary v_1, v_2. By (4) and by associativity $\overline{\overline{\overline{v_1} \cup v_2}} = \overline{(v_1 \cup v_2) \cup \overline{\overline{v_1} \cup v_2}} \cup \overline{\overline{v_2}} = \overline{v_1 \cup v_2 \cup \overline{\overline{v_1} \cup v_2}} \cup \overline{\overline{v_2}}$. Hence by involution $\overline{v_1 \cup v_2 \cup \overline{\overline{v_1} \cup v_2}} \cup v_2 = \overline{\overline{\overline{v_1} \cup v_2}} = \overline{v_1} \cup v_2$. This completes the proof of (5).

Because of commutativity and by associativity

$$\forall(x_1, x_2, x_3)\ x_3 \cup x_1 \cup x_2 = x_1 \cup x_2 \cup x_3. \tag{6}$$

Because of idempotency and by associativity

$$\forall(x_1, x_2)\ x_1 \cup x_1 \cup x_2 = x_1 \cup x_2. \tag{7}$$

We show directly that

$$\forall(x_1, x_2)\ \overline{x_1 \cup x_2} \cup \overline{x_2} = \overline{x_2}. \tag{8}$$

Let us consider arbitrary v_1, v_2. By (4) and by (7) $\overline{v_1 \cup v_2} \cup \overline{v_2} = \overline{v_1 \cup v_2} \cup \overline{\overline{v_1} \cup v_2} = \overline{v_2}$. This completes the proof of (8).

We show directly that

$$\forall(x_1, x_2)\ \overline{x_1 \cup \overline{x_2}} \cup x_2 = x_2. \tag{9}$$

Let us consider arbitrary v_1, v_2. By (8) and by involution $\overline{v_2 \cup \overline{v_1}} \cup v_1 = \overline{\overline{v_1}} = v_1$. This completes the proof of (9).

Hence by commutativity $\forall(x_1, x_2)\ \overline{\overline{x_2} \cup x_1} \cup x_2 = x_2$. Hence by (6)

$$\forall(x_1, x_2, x_3)\ x_3 \cup x_2 \cup \overline{\overline{x_3} \cup x_1} = x_2 \cup x_3. \tag{10}$$

Hence by (5)

$$\forall(x_1, x_2)\ \overline{\overline{x_2} \cup x_1} \cup x_2 = \overline{x_1} \cup x_2. \tag{11}$$

By (7) and by commutativity

$$\forall(x_1, x_2)\ x_1 \cup x_2 \cup x_1 = x_1 \cup x_2. \tag{12}$$

We show directly that

$$\forall(x_1, x_2)\ \overline{x_1 \cup x_2} \cup x_2 = \overline{x_1} \cup x_2. \tag{13}$$

Let us consider arbitrary v_1, v_2. By (12) and by (11) $\overline{v_1 \cup v_2} \cup v_2 = \overline{v_2 \cup v_1} \cup v_2 = \overline{v_1} \cup v_2$. This completes the proof of (13).

By (9) and by (6)

$$\forall(x_1, x_2, x_3)\ x_3 \cup x_2 \cup \overline{x_1 \cup \overline{x_3}} = x_2 \cup x_3. \tag{14}$$

By (4) and by commutativity

$$\forall(x_1, x_2)\ \overline{x_2 \cup x_1} \cup \overline{\overline{x_1} \cup x_2} = \overline{x_2}. \tag{15}$$

We show directly that

$$\forall(x_1, x_2, x_3)\ \overline{x_2 \cup \overline{x_1 \cup x_3}} = \overline{x_2 \cup \overline{x_1} \cup x_3} \cup \overline{x_2 \cup \overline{x_3}}. \tag{16}$$

Let us consider arbitrary v_1, v_2, v_3. By (14), (15), by involution, and by associativity

$$\begin{aligned} \overline{v_1 \cup \overline{v_2 \cup \overline{\overline{v_3}}}} &= \overline{\overline{(v_1 \cup \overline{v_2 \cup \overline{\overline{v_3}}}) \cup v_3} \cup \overline{v_1 \cup \overline{v_3}}} \\ &= \overline{\overline{(v_1 \cup \overline{v_2 \cup v_3})} \cup v_3 \cup \overline{v_1 \cup \overline{v_3}}} \\ \overline{v_1 \cup \overline{v_2 \cup \overline{\overline{v_3}}}} &= \overline{\overline{v_1 \cup \overline{v_2 \cup v_3} \cup v_3} \cup \overline{v_1 \cup \overline{v_3}}}. \end{aligned}$$

Hence by (13) and by involution $\overline{v_1 \cup \overline{v_2 \cup v_3} \cup v_3} \cup \overline{v_1 \cup \overline{v_3}} = \overline{v_1 \cup \overline{v_2 \cup \overline{\overline{v_3}}}} = \overline{v_1 \cup \overline{v_2 \cup v_3}}$. This completes the proof of (16).

Hence by (3)

$$\neg\, \overline{\overline{a} \cup \overline{c} \cup b} \cup \overline{\overline{a} \cup \overline{b}} = \overline{\overline{a} \cup \overline{b}} \cup \overline{\overline{a} \cup \overline{c}}. \tag{17}$$

By (8) and by commutativity

$$\forall(x_1, x_2)\ \overline{x_2} \cup \overline{x_1 \cup x_2} = \overline{x_2}. \tag{18}$$

We show directly that

$$\forall(x_1, x_2, x_3)\ \overline{\overline{x_2 \cup x_1 \cup x_3} \cup \overline{x_2 \cup \overline{x_3}}} \cup \overline{x_2 \cup x_1} = \overline{x_2 \cup x_1 \cup x_3} \cup \overline{x_2 \cup \overline{x_3}}. \tag{19}$$

Let us consider arbitrary v_1, v_2, v_3. By (10), (18), (16), and by involution

$$\begin{aligned} \overline{v_1 \cup \overline{\overline{v_3} \cup v_2}} &= \overline{\overline{v_1 \cup \overline{\overline{v_3} \cup v_2}} \cup \overline{v_1 \cup v_3}} \\ &= \overline{(\overline{v_1 \cup \overline{\overline{v_3}} \cup v_2} \cup \overline{v_1 \cup \overline{v_2}}) \cup \overline{v_1 \cup v_3}} \\ \overline{v_1 \cup \overline{\overline{v_3} \cup v_2}} &= \overline{(\overline{v_1 \cup v_3 \cup v_2} \cup \overline{v_1 \cup \overline{v_2}}) \cup \overline{v_1 \cup v_3}}. \end{aligned}$$

Hence by associativity, (16), and by involution

$$\begin{aligned} \overline{\overline{v_1 \cup v_3 \cup v_2} \cup \overline{v_1 \cup \overline{v_2}} \cup \overline{v_1 \cup v_3}} &= \overline{v_1 \cup \overline{\overline{v_3} \cup v_2}} \\ &= \overline{v_1 \cup \overline{\overline{v_3} \cup v_2}} \cup \overline{v_1 \cup \overline{v_2}} \\ \overline{\overline{v_1 \cup v_3 \cup v_2} \cup \overline{v_1 \cup \overline{v_2}} \cup \overline{v_1 \cup v_3}} &= \overline{v_1 \cup v_3 \cup v_2} \cup \overline{v_1 \cup \overline{v_2}}. \end{aligned}$$

This completes the proof of (19).

We show directly that

$$\forall(x_1, x_2, x_3)\ x_1 \cup x_2 \cup x_1 \cup x_3 = x_1 \cup x_2 \cup x_3. \tag{20}$$

Let us consider arbitrary v_1, v_2, v_3. By (12) and by associativity $(v_2 \cup v_1) \cup v_3 = v_2 \cup (v_1 \cup v_2) \cup v_3$. Hence by associativity $v_2 \cup v_1 \cup v_3 = v_2 \cup (v_1 \cup v_2) \cup v_3 = v_2 \cup v_1 \cup v_2 \cup v_3$. This completes the proof of (20).

By (6) and by (4)

$$\forall(x_1, x_2, x_3)\ \overline{x_2 \cup x_3 \cup x_1} \cup \overline{\overline{x_1} \cup x_2 \cup x_3} = \overline{x_2 \cup x_3}. \tag{21}$$

Therefore

$$\forall(x_1, x_2, x_3, x_4)\ \overline{x_3 \cup x_4 \cup x_1} \cup x_2 \cup \overline{\overline{x_1} \cup x_3 \cup x_4} = \overline{x_3 \cup x_4 \cup x_1} \cup x_2 \cup \overline{x_3 \cup x_4}. \tag{22}$$

We show directly that

$$\forall(x_1, x_2, x_3)\ x_2 \cup x_1 \cup x_3 = x_1 \cup x_2 \cup x_3. \tag{23}$$

Let us consider arbitrary v_1, v_2, v_3. Because of commutativity and by associativity $v_1 \cup v_2 \cup v_3 = (v_2 \cup v_1) \cup v_3 = v_2 \cup v_1 \cup v_3$. This completes the proof of (23).

We show directly that

$$\forall(x_1, x_2, x_3, x_4)\ \overline{x_3 \cup x_4 \cup x_1} \cup x_2 \cup \overline{x_3 \cup x_4} = x_2 \cup \overline{x_3 \cup x_4}. \tag{24}$$

Let us consider arbitrary v_1, v_2, v_3, v_4. By (21), (23), and by (22) $v_4 \cup \overline{v_1 \cup v_2} = \overline{v_1 \cup v_2 \cup v_3} \cup v_4 \cup \overline{\overline{v_3} \cup v_1 \cup v_2} = \overline{v_1 \cup v_2 \cup v_3} \cup v_4 \cup \overline{v_1 \cup v_2}$. This completes the proof of (24).

Hence by (19) $\forall(x_1, x_2, x_3)\ \overline{x_2 \cup \overline{x_3}} \cup \overline{x_2 \cup x_1} = \overline{x_2 \cup x_1 \cup x_3} \cup \overline{x_2 \cup \overline{x_3}}$. This contradicts (17). Thus we have completed the proof of (1).

q.e.d.