

Name Isaac Newton

1. Evaluate  $\int (\theta^2 + 1) \cos \theta d\theta$ .

$$\int (\theta^2 + 1) \cos \theta d\theta = \int \theta^2 \cos \theta d\theta + \int \cos \theta d\theta$$

To find  $\int \theta^2 \cos \theta d\theta$ , let

$$u = \theta^2, \quad dv = \cos \theta d\theta$$

$$du = 2\theta d\theta, \quad v = \sin \theta, \quad \text{so}$$

$$\int \theta^2 \cos \theta d\theta = \theta^2 \sin \theta - 2 \int \theta \sin \theta d\theta.$$

To find  $\int \theta \sin \theta d\theta$ , let

$$u = \theta, \quad dv = \sin \theta d\theta$$

$$du = d\theta, \quad v = -\cos \theta, \quad \text{so}$$

$$\int \theta \sin \theta d\theta = -\theta \cos \theta + \int \cos \theta d\theta = -\theta \cos \theta + \sin \theta + C.$$

Putting all this together, we find

$$\begin{aligned} \int (\theta^2 + 1) \cos \theta d\theta &= \theta^2 \sin \theta - 2(-\theta \cos \theta + \sin \theta) + \sin \theta + C \\ &= \boxed{\theta^2 \sin \theta + 2\theta \cos \theta - \sin \theta + C} \end{aligned}$$

(This problem can also be done by tabular integration.)

Name \_\_\_\_\_

2. Evaluate  $\int \frac{4xe^{x^2}}{e^{2x^2} + 2e^{x^2} + 2} dx.$

Let  $y = e^{x^2}$ , so  $dy = 2x e^{x^2}$  and

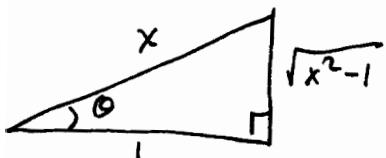
$$\begin{aligned}\int \frac{4x e^{x^2}}{e^{2x^2} + 2e^{x^2} + 2} dx &= \int \frac{2 dy}{y^2 + 2y + 2} \\&= \int \frac{2 dy}{(y+1)^2 + 1} \\&= 2 \tan^{-1}(y+1) + C \quad (\text{since } d(y+1)=dy) \\&= 2 \tan^{-1}(e^{x^2} + 1) + C.\end{aligned}$$

Name \_\_\_\_\_

3. Evaluate  $\int \frac{\sqrt{x^2 - 1}}{x^2} dx$ . Hint: you may at some point want to use  $\sin^2 \theta = 1 - \cos^2 \theta$ .

Let  $x = \sec \theta$ , so  $dx = \sec \theta \tan \theta d\theta$  and  
 $\sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$ . Then

$$\begin{aligned} \int \frac{\sqrt{x^2 - 1}}{x^2} dx &= \int \frac{\tan \theta \sec \theta \tan \theta d\theta}{\sec^2 \theta} \\ &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta \\ &= \int \frac{\sin^2 \theta \cos \theta}{\cos^2 \theta} d\theta \\ &= \int \frac{\sin^2 \theta}{\cos \theta} d\theta \\ &= \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta \\ &= \int \sec \theta d\theta - \int \cos \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C \end{aligned}$$



$$= \ln |x + \sqrt{x^2 - 1}| - \frac{\sqrt{x^2 - 1}}{x} + C.$$

Name \_\_\_\_\_

4. Does  $\int_0^\infty \frac{\sin^2 x}{x^2} dx$  converge or diverge? Give your reasons.

This integral (converges). As  $x \rightarrow 0$ ,  
 $\frac{\sin x}{x} \rightarrow 1$ , so  $\frac{\sin^2 x}{x^2} \rightarrow 1$  and  $\int_0^1 \frac{\sin^2 x}{x^2} dx$   
is actually a proper integral; the  
only bad point is  $\infty$ .

Now  $|\sin x| \leq 1$  for all  $x$ , so  
 $\int_1^\infty \left| \frac{\sin^2 x}{x^2} \right| dx \leq \int_1^\infty \frac{1}{x^2} dx$  and this integral  
converges by the p-test at  $\infty$  ( $p=2 > 1$ ),  
so  $\int_1^\infty \frac{\sin^2 x}{x^2} dx$  converges by absolute comparison.

Name \_\_\_\_\_

5. For each of the following three series, state whether it converges or diverges and give your reasons.

a)  $\sum_{n=0}^{\infty} \frac{7^n - 2^n}{(2n)!}$ . This series **(converges)**.

Apply the ratio test to  $\sum_{n=0}^{\infty} \frac{7^n}{(2n)!}$ . We have

$$\frac{7^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{7^n} = \frac{7}{(2n+2)(2n+1)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since  $p = 0 < 1$ , this series converges. Then  $\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$  also converges by comparison, so the original series converges.

b)  $\sum_{n=1}^{\infty} \frac{n}{n^2 + \sqrt{n}}$ . This series **(diverges)**.

As  $n \rightarrow \infty$ ,  $\frac{n}{n^2 + \sqrt{n}} \sim \frac{n}{n^2} = \frac{1}{n}$ . But  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges by the p-test, so our series diverges by limit comparison.

c)  $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{2^n + n^2}$ . This series **(diverges)** by the  $n^{\text{th}}$  term test, since the sequence  $(-1)^n \frac{2^n}{2^n + n^2}$  does not converge to 0.

Name \_\_\_\_\_

6. For what values of  $x$  does each of the following two series converge? Give your reasons.

a)  $\sum_{n=1}^{\infty} \frac{(x+3)^n}{\sqrt{n^3}}$ . This converges for  $-4 \leq x \leq -2$  and diverges elsewhere. Apply the root test:

$$\left( \frac{|x+3|^n}{\sqrt{n^3}} \right)^{\frac{1}{n}} = \frac{|x+3|}{(n^{\frac{1}{n}})^3} \rightarrow |x+3| \text{ as } n \rightarrow \infty,$$

so we have convergence at least for  $|x+3| < 1$  (i.e.,  $-4 < x < -2$ ) and divergence for  $|x+3| > 1$ .

At the end points we have  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ , each of which converges absolutely by the p-test ( $p = \frac{3}{2} > 1$ ).

b)  $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n}$ . This converges for  $0 \leq x < 1$  and diverges elsewhere. Apply the root test:

$$\left( \frac{|2x-1|^n}{n} \right)^{\frac{1}{n}} = \frac{|2x-1|}{n^{\frac{1}{n}}} \rightarrow |2x-1| \text{ as } n \rightarrow \infty,$$

so we have convergence at least for  $|2x-1| < 1$  (i.e., for  $|x - \frac{1}{2}| < \frac{1}{2}$  — i.e., for  $0 < x < 1$ ) and divergence for  $|2x-1| > 1$ . At the endpoints, for  $x=0$  we have  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ , which converges by the alternating series test, and for  $x=1$  we have  $\sum_{n=1}^{\infty} \frac{1}{n}$ , which diverges by the p-test.

Name \_\_\_\_\_

7. Find the second order Taylor polynomial of  $\tan^{-1} x$  about the center  $a = \frac{1}{2}$ .

Make a table:

$n$	$f^{(n)}(x)$	$f^{(n)}(a)$	$f^{(n)}(a)/n!$
0	$\tan^{-1} x$	$\tan^{-1} \frac{1}{2}$	$\tan^{-1} \frac{1}{2}$
1	$\frac{1}{1+x^2}$	$\frac{4}{5}$	$\frac{4}{5}$
2	$-\frac{2x}{(1+x^2)^2}$	$-\frac{16}{25}$	$-\frac{8}{25}$

Thus the second order Taylor polynomial is

$$\tan^{-1} \frac{1}{2} + \frac{4}{5} \left(x - \frac{1}{2}\right) - \frac{8}{25} \left(x - \frac{1}{2}\right)^2.$$

Name \_\_\_\_\_

8. Find  $\sqrt[3]{1.01}$  with an error at most 0.0001. Hint:  $\sqrt[3]{1.01} = (1 + 0.01)^{1/3}$ .

Let  $f(x) = (1+x)^{\frac{1}{3}}$  and  $a=0$ , and make a table:

$n$	$f^{(n)}(x)$	$f^{(n)}(a)$	$f^{(n)}(a)/n!$
0	$(1+x)^{\frac{1}{3}}$	1	.1
1	$\frac{1}{3}(1+x)^{-\frac{2}{3}}$	$\frac{1}{3}$	$\frac{1}{3}$
2	$-\frac{2}{9}(1+x)^{-\frac{5}{3}}$		

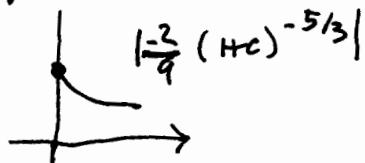
By Taylor's theorem,

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x - \frac{2}{9}(1+c)^{-\frac{5}{3}}x^2 \text{ for some } c$$

between 0 and  $x$ . Apply this to  $x=0.01$ :

$$\sqrt[3]{1.01} = 1 + \frac{1}{3}0.01 - \frac{2}{9}(1+c)^{-\frac{5}{3}}0.0001$$

for some  $c$  with  $0 \leq c \leq 0.01$ . The worst case is  $c=0$ :



and  $|- \frac{2}{9}0.0001| < 0.0001$ , as required. So

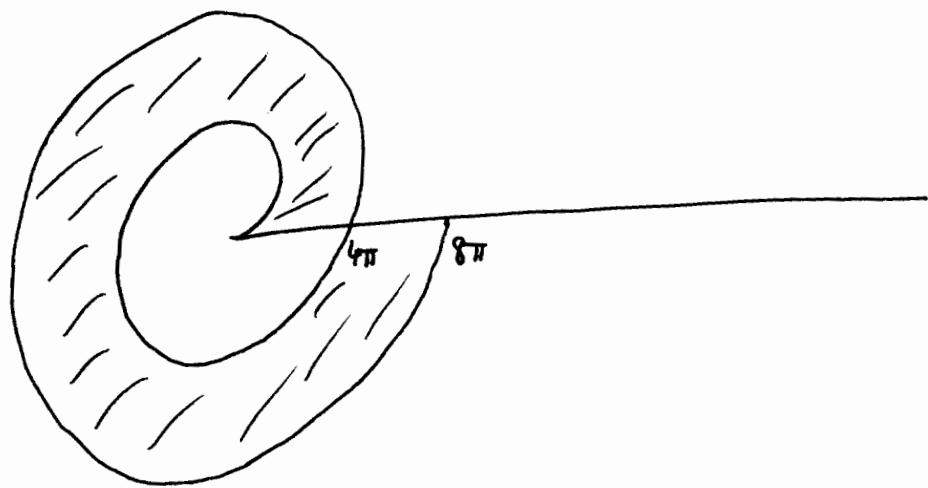
$$\sqrt[3]{1.01} \approx 1 + \frac{1}{3}0.01 = 1.0033 \quad \text{with an error}$$

at most 0.0001.

Name \_\_\_\_\_

9.

- a) Draw the graph of the first two turns of the spiral given in polar coordinates by  $r = 2\theta$  (that is, for  $0 \leq \theta \leq 4\pi$ ).



- b) Find the area of the region enclosed between the first and second turn of the spiral (i.e., the region between the curves  $r = 2\theta$  for  $0 \leq \theta \leq 2\pi$  and  $r = 2\theta$  for  $2\pi \leq \theta \leq 4\pi$ , as well as the positive  $x$ -axis between 0 and  $4\pi$ ).

$$\begin{aligned}
 A &= \int_0^{4\pi} \frac{1}{2} (2\theta)^2 d\theta - \int_0^{2\pi} \frac{1}{2} (2\theta)^2 d\theta \\
 &= \int_{2\pi}^{4\pi} \frac{1}{2} (2\theta)^2 d\theta \\
 &= 2 \int_{2\pi}^{4\pi} \theta^3 d\theta \\
 &= \frac{2}{3} \theta^3 \Big|_{2\pi}^{4\pi} = \frac{2}{3} \cdot 64\pi^3 - \frac{2}{3} \cdot 8\pi^3 = \frac{2}{3} \cdot 56\pi^3 \\
 &= \left( \frac{112}{3} \pi^3 \right).
 \end{aligned}$$

Name \_\_\_\_\_

10. Find all real or complex solutions to  $z^8 - z^4 - 2 = 0$ .

This is a quadratic equation for  $z^4$ :

$$(z^4)^2 - (z^4) - 2 = 0,$$

so solve first for  $z^4$ , finding

$$z^4 = \frac{1 \pm \sqrt{1+8}}{2} = 2, -1.$$

Now find all fourth roots of 2. The obvious one is  $2^{\frac{1}{4}}$ . Multiply this by the four 4th roots of 1 (namely, 1,  $i$ ,  $-1$ ,  $-i$ ) to obtain  $(2^{\frac{1}{4}}, 2^{\frac{1}{4}}i, -2^{\frac{1}{4}}, -2^{\frac{1}{4}}i)$ .

Next find all fourth roots of  $-1$ . In polar form,  $-1 = e^{i\pi}$ . The obvious fourth root is  $e^{i\frac{\pi}{4}} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ . Multiply by the four 4th roots of 1 to obtain

$$\left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right).$$

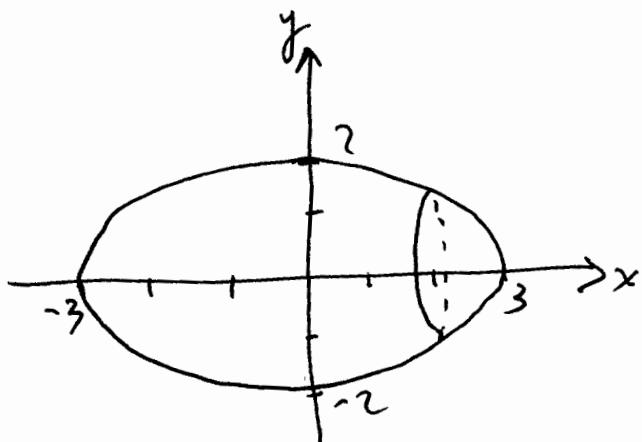
The circled numbers are the eight solutions to  $z^8 - z^4 - 2 = 0$ .

Name \_\_\_\_\_

11. The region inside the curve

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and above the  $x$ -axis is revolved about the  $x$ -axis. Find the volume.



$$\begin{aligned}
 V &= \int_{-3}^3 \pi y^2 dx = \int_{-3}^3 \pi 4\left(1 - \frac{x^2}{9}\right) dx \\
 &= 2 \int_0^3 \pi 4\left(1 - \frac{x^2}{9}\right) dx \\
 &= 8\pi \int_0^3 dx - \frac{8\pi}{9} \int_0^3 x^2 dx \\
 &= 24\pi - \frac{8\pi}{9} \left. \frac{x^3}{3} \right|_0^3 \\
 &= 24\pi - \frac{8\pi}{9} \cdot 9 = \boxed{16\pi}.
 \end{aligned}$$

Name \_\_\_\_\_

12. Solve the initial value problem

$$x \frac{dy}{dx} - 2y = x^3 e^x, \quad y(1) = 0.$$

Divide by  $x$ :

$$\frac{dy}{dx} - \frac{2}{x} y = x^2 e^x$$

This is in standard form with  $P(x) = -\frac{2}{x}$ ,  $Q(x) = x^2 e^x$ .

The integrating factor is

$$v(x) = e^{\int P(x) dx} = e^{\int -\frac{2}{x} dx} = e^{-2 \ln|x|} = |x|^{-2} = \frac{1}{x^2}.$$

The general solution is

$$\begin{aligned} y &= \frac{1}{v(x)} \int v(x) Q(x) dx \\ &= x^2 \int \frac{1}{x^2} (x^2 e^x) dx = x^2 \int e^x dx = x^2 (e^x + C). \end{aligned}$$

We are given  $y(1) = 0$ , so

$$0 = 1^2 (e^1 + C), \text{ and } C = -e.$$

The solution is

$$y = x^2 (e^x - e).$$