Proof of the uniform convexity lemma

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Fix $1 < p < 2$. Our goal is to prove that for every $a, b \in L_p$,

$$\|a+b\|_p^2 + (p-1)\|a-b\|_p^2 \le 2(\|a\|_p^2 + \|b\|_p^2). \tag{1}$$

We require the following numerical lemma:

Lemma 0.1. *For $0 \le r \le 1$ define*

$$\alpha(r) = (1+r)^{p-1} + (1-r)^{p-1} \quad \text{and} \quad \beta(r) = \frac{(1+r)^{p-1} - (1-r)^{p-1}}{r^{p-1}}.$$

Then for every $A, B \in \mathbb{R}$,

$$\alpha(r)|A|^p + \beta(r)|B|^p \le |A+B|^p + |A-B|^p.$$

Proof. We may clearly assume that $A, B > 0$. Observe first of all that $\beta(r) \le \alpha(r)$ for all $r \in [0,1]$. Indeed, setting $h(r) = \alpha(r) - \beta(r)$ we have $h(1) = 0$ and

$$h'(r) = -(p-1)\left(\frac{1}{r^p} + 1\right)\left[\frac{1}{(1-r)^{2-p}} - \frac{1}{(1+r)^{2-p}}\right] \le 0.$$

It follows that if $0 < A < B$ then $\alpha(r)A^p + \beta(r)B^p \le \alpha(r)B^p + \beta(r)A^p$, which implies that it enough to prove that for $0 < B < A$, $\alpha(r)A^p + \beta(r)B^p \le (A+B)^p + (A-B)^p$. Dividing by A^p, it suffices to show that for $0 \le R \le 1$, the function $F(r) = \alpha(r) + R^p\beta(r)$ achieves its global maximum at $r = R$. But

$$F'(r) = (p-1)[(1+r)^{p-2} - (1-r)^{p-2}]\left[1 - \left(\frac{R}{r}\right)^p\right],$$

Thus, the only point in $(0,1)$ at which F' vanishes is $r = R$, and since $1 < p < 2$, $F'(1)$ is negative. This implies that F is maximal at $r = R$. □

Corollary 0.2 (Hanner's inequality for $1 \le p \le 2$)**.** *For every $f, g \in L_p$,*

$$\big|\|f\|_p - \|g\|_p\big|^p + (\|f\|_p + \|g\|_p)^p \le \|f+g\|_p^p + \|f-g\|_p^p.$$

Proof. By symmetry we may assume that $r = \|g\|_p/\|f\|_p \le 1$. By Lemma 0.1 the following pointwise inequality holds:

$$\alpha(r)|f|^p + \beta(r)|g|^p \le |f+g|^p + |f-g|^p.$$

Integrating and simplifying gives the required result. □

The following numerical lemma is well known.

Lemma 0.3 (Beckner's two-point inequality)**.** *For every* $a, b \in \mathbb{R}$*,*

$$[a^2 + (p-1)b^2]^{1/2} \leq \left(\frac{|a+b|^p + |a-b|^p}{2}\right)^{1/p}.$$

Proof. If $|a| < |b|$ then since $p < 2$, $a^2 + (p-1)b^2 \leq b^2 + (p-1)a^2$. We may therefore assume that $|a| \geq |b| > 0$. Set $x = b/a$. Our goal is to show that $[1 + (p-1)x^2]^{p/2} \leq \frac{(1+x)^p + (1-x)^p}{2}$ for every $x \in [-1, 1]$. Now, $\frac{(1+x)^p + (1-x)^p}{2} = \sum_{k=0}^{\infty} \binom{p}{2k} x^{2k} \geq 1 + \frac{p(p-1)}{2} x^2$, where we have used the fact that since $p < 2$, $\binom{p}{2k} \geq 0$. Finally, the inequality $1 + \frac{p(p-1)}{2} x^2 \geq [1 + (p-1)x^2]^{p/2}$ follows from the elementary fact that $(1+t)^\alpha \leq 1 + \alpha t$ for every $t, \alpha \in [0, 1]$. □

Let's complete the proof of (1). Fix $x, y \in L_p$. Then

$$\begin{aligned}
\left(\frac{\|x+y\|_p^2 + \|x-y\|_p^2}{2}\right)^{1/2} &\geq \left(\frac{\|x+y\|_p^p + \|x-y\|_p^p}{2}\right)^{1/p} && \text{(since } p \leq 2\text{)} \\
&\geq \left[\frac{(\|x\|_p + \|y\|_p)^p + \big|\|x\|_p - \|y\|_p\big|^p}{2}\right]^{1/p} && \text{(Hanner's inequality)} \\
&\geq [\|x\|_p^2 + (p-1)\|y\|_p^2]^{1/2} && \text{(Beckner's inequality),}
\end{aligned}$$

and this is equivalent to (1) (setting $a = x + y$ and $b = x - y$).