Metric Cotype*

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Abstract

We introduce the notion of *metric cotype*, a property of metric spaces related to a property of normed spaces, called *Rademacher cotype*. Apart from settling a long standing open problem in metric geometry, this property is used to prove the following dichotomy: A family of metric spaces $\mathcal{F}$ is either almost universal (i.e., contains any finite metric space with any distortion > 1), or there exists $\alpha > 0$, and arbitrarily large n-point metrics whose distortion when embedded in any member of $\mathcal{F}$ is at least $\Omega\left((\log n)^{\alpha}\right)$. The same property is also used to prove strong non-embeddability theorems of L_q into L_p, when $q > \max\{2, p\}$. Finally we use metric cotype to obtain a new type of isoperimetric inequality on the discrete torus.

1 Introduction

1.1 An Embedding Dichotomy In the past decade the theory of finite metric spaces has become an intensively investigated topic in the theoretical computer science literature due to its remarkable applicability to algorithm design.

One approach in this vein is to reduce optimization problems over general metric spaces to a class of "special" metrics which has more structure (e.g., convex combination of tree metrics [1, 5]), and solve the optimization problem over the class of special metrics. The class of special metric spaces is chosen to balance between the structure needed for developing an algorithmic solution, and the "distance" of the special metrics from the original metric. That "distance" influences the quality of the algorithmic solution in the original metric.

A useful measure for the "distance"' between metric spaces is the distortion.

DEFINITION 1.1. *Given two metric spaces $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$, and an injective mapping $f : \mathcal{M} \hookrightarrow \mathcal{N}$, we denote the* distortion *of f by*

$$\mathrm{dist}(f) := \sup_{\substack{x,y\in\mathcal{M}\\ x\neq y}} \frac{d_{\mathcal{N}}(f(x), f(y))}{d_{\mathcal{N}}(x, y)} \cdot \sup_{\substack{x,y\in\mathcal{M}\\ x\neq y}} \frac{d_{\mathcal{M}}(x, y)}{d_{\mathcal{M}}(f(x), f(y))}.$$

The smallest such distortion is denoted $c_{\mathcal{N}}(\mathcal{M})$, i.e. $c_{\mathcal{N}}(\mathcal{M}) := \inf\{\mathrm{dist}(f) : \ f : \mathcal{M} \hookrightarrow \mathcal{N}\}$.

Bourgain's embedding theorem [8] and Bartal's probabilistic embedding theorem [5, 16] established Hilbert spaces, ℓ_1, and and convex combination of tree metrics as useful host spaces for which the distortion of embedding n-point metrics is $O(\log n)$.

It is therefore interesting to find out whether this approach can gives distortions which are significantly better than the guarantee in Bourgain's theorem. A concrete natural question in this vein is:

> Is there a non-trivial class of metric spaces $\mathcal{N}$ for which $c_{\mathcal{N}}(X)$ is significantly less than $\log |X|$, for every finite metric space X?

Motivated by related questions, Arora, Lovász, Newman, Rabani, Rabinovich and Vempala [3] proved the following dichotomy.

THEOREM 1.1. *For any class $\mathcal{F}$ of metric spaces,*

1. *Either $\mathcal{F}$ is almost universal, i.e., for any finite metric space M, $c_{\mathcal{F}}(M) = 1$, or*
2. *for every $\gamma \geq 1$, there exists a finite metric space M such that $c_{\mathcal{F}}(M) > \gamma$.*

The proof is a simple corollary of Matoušek's bounded-distortion (BD) Ramsey theorem [29]. Arora et. al. conjecture that a stronger form of Theorem 1.1 is true for *normal classes of metrics*.[1]

CONJECTURE 1.1. *Let $\mathcal{F}$ be a* normal *metric class which does not contain all finite metrics with distortion arbitrarily close to* 1*. Then there exists $\alpha > 0$ and arbitrarily large n-point metric spaces $\mathcal{M}_n$ such that $c_{\mathcal{F}}(\mathcal{M}_n) \geq (\log n)^{\alpha}$.*

Arora et. al. also give some supporting evidence for Conjecture 1.1 to be true. Here we settle this conjecture positively, without any restriction on the class $\mathcal{F}$. Namely, we prove

THEOREM 1.2. *For any class $\mathcal{F}$ of metric spaces:*

1. *Either $\mathcal{F}$ is almost universal, or*

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*Extended abstract. A full version of this paper with all the details is available at `http://arxiv.org/math.FA/0506201`

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[1]We refer to [3] for the definition of a normal class of metrics, since we will not use this notion in what follows.

2. *there exists $\alpha > 0$ and a sequence of metric spaces $\{M_n\}_{n\ge 1}$, such that $|M_n| = n$, and $c_{\mathcal{F}}(M_n) = \Omega\left((\log n)^{\alpha}\right)$.*

For Hilbert space H, $\sup\{c_H(M) : |M| = n\} = \Theta(\log n)$ [8, 26]. We do not know whether there exists a class of metric spaces $\mathcal{F}$ which is not almost universal, but for which $\sup\{c_{\mathcal{F}}(M) : |M| = n\} = O\left((\log n)^{\beta}\right)$, for some $\beta \in (0,1)$.

Theorem 1.2 is proved using a newly introduced property of metric spaces called *metric cotype*. Its origins come from Banach space theory, which we review next.

1.2 Banach Space's Perspective The parallelogram equality states that in Hilbert space H, for any $x, y \in H$,

$$\forall x, y \in H \qquad \|x-y\|^2 + \|x+y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

It turns out that this equality characterizes Hilbert space (see [2]). For various reasons which will become clearer in the sequel, researchers in Banach space theory generalized this property into two "isomorphic" inequalities known today as (Rademacher) type and cotype.

A Banach space X is said to have (Rademacher) type $p > 0$ if there exists a constant $T < \infty$ such that for every n and every $x_1, \ldots, x_n \in X$,

$$\mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^n} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_X^p \le T^p \sum_{j=1}^n \|x_j\|_X^p. \tag{1.1}$$

X is said to have (Rademacher) cotype $q > 0$ if there exists a constant $C < \infty$ such that for every n and every $x_1, \ldots, x_n \in X$,

$$\mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^n} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_X^q \ge \frac{1}{C^q} \sum_{j=1}^n \|x_j\|_X^q. \tag{1.2}$$

The infimum over C satisfying (1.2) for any $n \in \mathbb{N}$, and $x_1, \ldots, x_n \in X$ is denoted $C_q(X)$.

The notions of type and cotype of Banach spaces are the basis of a deep and rich theory which encompasses diverse aspects of the local theory of Banach spaces. We refer to the full version of this paper for references on these topics. Here we mention only few highlights of this theory:

1. Kwapien's Theorem [24] generalizes the isometric characterization of Hilbert space into an isomorphic one: A Banach space X is isomorphic (i.e., has a linear bijection with finite distortion) to Hilbert space if and only if it has type 2 and cotype 2.

2. Denote by p_X the supremum over p such that X has type p, and by by q_X the infimum over q such that X has cotype q. The Maurey-Pisier theorem [35, 32] states that for any $n \in \mathbb{N}$, and any $\eta > 0$, X linearly contains copies of $\ell_{q_X}^n$ and $\ell_{p_X}^n$ with distortion at most $1+\eta$.

3. Dvoretzky's theorem (see [31, Chap. 14]) states that for any $\eta > 0$ and $n \in \mathbb{N}$, any n-dimensional normed space X contains a d dimensional linear subspace $Y \subseteq X$ that is isomorphic to ℓ_2^d with distortion $1+\eta$, where $d = \Omega_\eta(\log n)$. The logarithmic estimate on d is known to be asymptotically tight. However, Figiel, Lindenstrauss, and Milman [17] have shown that it is possible find such Y which is $1+\eta$ isomorphic to ℓ_2^d, and $d = \Omega_{q_X,\eta}(n^{2/q_X})$.

The notions of type and cotype are clearly *linear* notions, since their definition involves addition and multiplication by scalars. However, in 1976 Ribe (see [7]) proved that if X and Y are uniformly homeomorphic Banach spaces (i.e., there exists a bijection f which is uniformly continuous and f^{-1} is also uniformly continuous) then X is finitely representable in Y, and vice versa (X is said to be finitely representable in Y if there exists a constant $K > 0$ such that any finite dimensional subspace of Y is K-isomorphic to a subspace of Y). This theorem suggests that "local properties" of Banach spaces, i.e. properties which are invariant under finite representability, have a purely metric characterization. Finding explicit manifestations of this phenomenon for specific local properties of Banach spaces (such as type, cotype and super-reflexivity), has long been a major driving force in the bi-Lipschitz theory of metric spaces (see [9] for a discussion of this research program). Once this is achieved, one could define the notion of type and cotype of a metric space, and then hopefully transfer some of the deep theory of type and cotype to the context of arbitrary metric spaces.

Enflo's pioneering work [12, 13, 14, 15] resulted in the formulation of a non-linear notion of type, known today as *Enflo type*. The basic idea is that given a Banach space X and $x_1, \ldots, x_n \in X$, one can consider the *linear* function $f : \{-1,1\}^n \to X$ given by $f(\varepsilon) = \sum_{j=1}^n \varepsilon_j x_j$. Then (1.1) becomes

$$\begin{aligned} &\mathbb{E}_\varepsilon \|f(\varepsilon) - f(-\varepsilon)\|_X^p \\ &\le T^p \sum_{j=1}^n \mathbb{E}_\varepsilon \| f(\varepsilon_1, \ldots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n) - \\ &\qquad f(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n)\|_X^p. \end{aligned} \tag{1.3}$$

One can thus say that a metric space $(\mathcal{M}, d_{\mathcal{M}})$ has Enflo type p if there exists a constant T such that for every $n \in \mathbb{N}$ and *every* $f : \{-1,1\}^n \to \mathcal{M}$,

$$\begin{aligned} &\mathbb{E}_\varepsilon\, d_{\mathcal{M}}(f(\varepsilon), f(-\varepsilon))^p \\ &\le T^p \sum_{j=1}^n \mathbb{E}_\varepsilon\, d_{\mathcal{M}}(f(\varepsilon_1, \ldots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n), \\ &\qquad f(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n))^p. \end{aligned} \tag{1.4}$$

There are two natural concerns about (1.4). First of all, while in the category of Banach spaces (1.4) is clearly a strengthening of (1.3) (as we are not restricting only to linear functions f), it isn't clear whether (1.4) follows from (1.3). Indeed, this problem was posed by Enflo in [15], and in full generality it remains open. Secondly, we do not know if (1.4) is a useful notion, in the sense that it yields metric variants of certain theorems from the linear theory of type. The first issue is addressed in [10, 36] where it is shown that for Banach spaces, Rademacher type p implies Enflo type p' for every $0 < p' < p$, and the same holds for a variant of Enflo type called BMW type. The second issue turned out not be problematic either: Enflo found striking applications of his notion of type to Hilbert's fifth problem in infinite dimensions [13, 14, 15], and to the uniform classification of L_p spaces [12]. Bourgain, Milman and Wolfson [10] obtained a non-linear version of the Maurey-Pisier theorem for type [35, 32] mentioned above, yielding a characterization of metric spaces which contain bi-Lipschitz copies of the Hamming cube. A stronger notion of non-linear type, known as Markov type, was introduced by Ball [4] in his study of the *Lipschitz extension problem*. This important notion has since found applications to various problems in metric geometry and computer science [27, 6, 34].

Despite the vast amount of research on non-linear type, a non-linear notion of cotype remained elusive. Indeed, the problem of finding a notion of cotype which makes sense for arbitrary metric spaces, and which coincides (or almost coincides) with the notion of Rademacher cotype when restricted to Banach spaces, became a central open problem in the field.

1.3 Metric Cotype In this paper we define a metric notion cotype.

DEFINITION 1.2. [**Metric cotype**] *Let $\mathcal{F}$ be a family of metric spaces and $q \geq p > 0$. We shall say $\mathcal{F}$ has metric cotype q with power p and with constant Γ if for every integer $n \in \mathbb{N}$, there exists an even integer m, such that for every $\mathcal{M} \in \mathcal{F}$ and every $f : \mathbb{Z}_m^n \to \mathcal{M}$,*

$$\sum_{j=1}^{n} \mathbb{E}_x \left[d_{\mathcal{M}} \left(f\left(x + \frac{m}{2} e_j\right), f(x) \right)^p \right] \leq \Gamma^p m^p n^{1-\frac{p}{q}}\, \mathbb{E}_{\varepsilon,x} \left[d_{\mathcal{M}}(f(x+\varepsilon), f(x))^p \right], \tag{1.5}$$

where the expectations above are taken with respect to uniformly chosen $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1, 0, 1\}^n$ (here, and in what follows we denote by $\{e_j\}_{j=1}^n$ the standard basis of $\mathbb{R}^n$). When $p = q$, we simply call the condition above metric cotype q. The smallest constant Γ with which inequality (1.5) *holds true is denoted $\Gamma_q^{(p)}(\mathcal{F})$. When $p = q$, we write $\Gamma_q(\mathcal{F}) = \Gamma_q^{(q)}(\mathcal{F})$. We further introduce the notation*

$$q^{(p)}(\mathcal{F}) = \inf\{q : \Gamma_q^{(p)}(\mathcal{F}) < \infty\}, \qquad \textit{and}$$
$$q(\mathcal{F}) = \inf\{q : \Gamma_q(\mathcal{F}) < \infty\}.$$

REMARK 1.1. *Metric cotype q is really a class of inequalities, depending on the power $p \in [1, q]$. In Banach spaces, these inequalities are provably closely related (see the full version). Their mutual relationship in general metric spaces is not yet understood. Different applications presented in this paper use different variants (power) of the metric cotype inequality.*

The following theorem is the main result of this paper:

THEOREM 1.3. *Let X be a Banach space, and $q \in [2, \infty)$. Then X has metric cotype q if and only if X has Rademacher cotype q. Moreover, $\frac{1}{2\pi} C_q(X) \leq \Gamma_q(X) \leq 90 C_q(X)$.*

Finding metric analogs of linear notions of normed spaces has proved to benefit the theory of finite metric spaces and algorithms. An example of this phenomenon is Theorem 1.2. That theorem actually follows from the following theorem, whose proof is discussed in Section 2.

THEOREM 1.4. *Let $\mathcal{F}$ be a family of metric spaces, then $q^{(2)}(\mathcal{F}) = \infty$ iff for any $m, n \in \mathbb{N}$, $c_{\mathcal{F}}([m]_\infty^n) = 1$, where $[m]_\infty^n$ is the grid $\{0, \ldots, m-1\}^n$ equipied with the ℓ_∞ norm.*

Theorem 1.4 can be viewed as a metric analog of a special case (cotype infinity) of the Maurey-Pisier Theorem mentioned above.

For perspective, we mention some previous examples of this interaction between Banach space theory and computer science:

I. Bourgain's famous embedding theorem [8] is motivated by John's theorem [22]. Bourgain's embedding technique has found many applications in computer science (see [21]).
II. Bourgain's work on the metric interpretation of superreflexivity [9] has been followed up by computer scientists regarding the embeddability of of tree metrics in Euclidean spaces [19, 28, 30, 20, 25].
III. Ball's notion of Markov type [4], partially motivated as another metric analog for type, has been used by the computer science community to analyze the Euclidean distortion of high-girth graphs, the Hamming cube, and their subsets [27, 6].
IV. Metric Ramsey theory (see [6] and references therein) is used to prove lower bound for some online optimization problems, and was partially motivated as a metric analog to Dvoretzky's theorem.

We hope that the present paper will serve as stimulus for further exportation of ideas from the highly developed theory of the geometry of Banach spaces to algorithmic research.

1.4 Other Applications We next consider various other consequences of the metric cotype property.

Relative conductance of $\mathbb{Z}_m^n$. Since L_q has cotype $\max\{2,q\}$, it also has metric cotype $\max\{2,q\}$. This fact is especially interesting for L_1. The metric cotype 2 with power 1 property of L_1 can be reinterpreted in terms of a new kind of a "relative conductance" property of $\mathbb{Z}_m^n$.

DEFINITION 1.3. *Let $G_1 = (V, E_1)$ be a d_1 regular graph and $G_2 = (V, E_2)$ be a d_2 regular graph, both on the same set of vertices, V. The conductance of G_2 relative to G_1 is defined as*

$$\Phi_{E_2/E_1}(V) = \min_{\emptyset \neq S \subseteq V} \frac{d_1 |E_2(S, \bar{S})|}{d_2 |E_1(S, \bar{S})|},$$

where for a graph $G = (V, E)$ and $A, B \subseteq V$, $E(A, B)$ denotes the edges in E which intersect both A and B.

Note that the usual conductance of a regular graph can be interpreted as a conductance relative to the complete graph (with self loops).

We next define two graphs on $\mathbb{Z}_m^n$:

$$E_1 = \{(x,y) \in V \times V : \exists j, x - y = \tfrac{m}{2} e_j \mod m\},$$
$$E_2 = \{(x,y) \in V \times V : x - y \in \{-1,0,1\}^n \mod m\}$$

The following theorem is implied by the cotype 2 with power 1 property of L_1. Its proof is discussed in Section 3.

THEOREM 1.5. *There exists universal constants $\beta \geq \alpha > 0$ such that for any $n \in \mathbb{N}$, and $m \in 4\mathbb{N}$,*

$$\beta \min\{\tfrac{\sqrt{n}}{m}, 1\} \geq \Phi_{E_2/E_1}(\mathbb{Z}_m^n) \geq \alpha \min\{\tfrac{\sqrt{n}}{m}, 1\}.$$

An interesting application of Theorem 1.5 is tight bounds on the embedding of ℓ_∞ grids in L_1.

COROLLARY 1.1. *Let $[m]_\infty^n$ denotes the grid $\{0, \ldots, m-1\}^n$ endowed with the ℓ_∞ norm, then $c_{L_1}([m]_\infty^n) = \Theta(\min\{m, \sqrt{n}\})$. The same holds for embedding into Hilbert space.*

Quadratic inequalities on the cut-cone. An intriguing aspect of Theorem 1.3 is that L_1 has metric cotype 2 with power 2. This inequality seems to be qualitatively stronger than the cotype 2 with power 1 property of L_1. It is a non-trivial inequality on L_1 which involves distances *squared*. To the best of our knowledge, all the known non-embeddability results for L_1 are based on Poincaré type inequalities in which distances are raised to the power 1. By the cut-cone representation of L_1 metrics (see [11]) it is enough to prove any such inequality for *cut metrics*, which are particularly simple. Theorem 1.3 seems to be the first truly "infinite dimensional" metric inequality in L_1. We believe that understanding such inequalities on L_1 deserves further scrutiny, especially as they hint at certain non-trivial (and non-linear) interactions between cuts.

Strong nonembeddability results for L_p. To state these results we need the following weak notion of distance respecting embedding due to Gromov [18].

DEFINITION 1.4. *Let $(\mathcal{M}, d_\mathcal{M})$ and $(\mathcal{N}, d_\mathcal{N})$ be metric spaces. A mapping $f : \mathcal{M} \to \mathcal{N}$ is called a* coarse embedding *if there exists two non-decreasing functions $\alpha, \beta : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} \alpha(t) = \infty$, and for every $x, y \in \mathcal{M}$, $\alpha(d_\mathcal{M}(x,y)) \leq d_\mathcal{N}(f(x), f(y)) \leq \beta(d_\mathcal{M}(x,y))$.*

In Section 4 we show:

THEOREM 1.6. *For any $r > \max\{2,q\}$, $q \geq 1$, ℓ_r does not coarsely embed in L_q.*

Theorem 1.6 generalizes a recent result of Johnson and Randrianarivony [23] who proved a special case of Theorem 1.6 when $p \in [1,2]$. This completes the coarse classification of L_p spaces since it is known [37, 33] that L_q coarsely embeds in L_p when $q \leq p$ or when $q \leq 2$.

Similar results hold for another type of weak embedding called *uniform embedding*. We will not discuss this topic here, and refer to the full version of this paper for more details.

2 Nonlinear Maurey-Pisier Theorem

In this section we sketch the proof of Theorem 1.4. The proof of Theorem 1.2 does not require much more than what is presented here, but due to space limitation will not be further discussed.

In what follows we denote by $\mathbf{diag}(\mathbb{Z}_m^n)$ the graph on $\mathbb{Z}_m^n$ in which $x, y \in \mathbb{Z}_m^n$ are adjacent if for every $i \in \{1, \ldots, n\}$, $x_i - y_i \in \{\pm 1 \mod m\}$.

DEFINITION 2.1. *Given $2 \leq q$, a family of metric space $\mathcal{F}$, an integer n and an even integer m, let $\Gamma_q^{(2)}(\mathcal{F}; n, m)$ be the infimum over all $\Gamma > 0$ such that for every $\mathcal{M} \in \mathcal{F}$, and every $f : \mathbb{Z}_m^n \to \mathcal{M}$,*

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_\mathcal{M}\left(f\left(x + \frac{m}{2} e_j\right), f(x)\right)^p d\mu(x) \tag{2.6}$$
$$\leq \Gamma^p m^p n^{1-\frac{p}{q}} \mathop{\mathbb{E}}_{\varepsilon \in \{\pm1,0\}^n} \int_{\mathbb{Z}_m^n} d_\mathcal{M}\left(f(x+\varepsilon), f(x)\right)^p d\mu(x).$$

With this notation,

$$\Gamma_q^{(2)}(\mathcal{F}) = \sup_{n \in \mathbb{N}} \Big(\inf_{m \in 2\mathbb{N}} \Gamma_q^{(2)}(\mathcal{F}; n, m) \Big).$$

For technical reasons that will become clear presently, given $\ell, n \in \mathbb{N}$ we denote by $\mathcal{B}(\mathcal{F}; n, \ell)$ the infimum over $\mathcal{B} > 0$ such that for every even $m \in \mathbb{N}$, every $\mathcal{M} \in \mathcal{F}$, and every $f : \mathbb{Z}_m^n \to \mathcal{M}$,

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}} \left(f\left(x+\ell e_j\right), f(x)\right)^2 d\mu(x) \le \mathcal{B}^2 \ell^2 n \mathop{\mathbb{E}}_{\varepsilon \in \{\pm 1\}^n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^2 d\mu(x). \tag{2.7}$$

Note that $\Gamma(\mathcal{F}; n, m)$, and $\mathcal{B}(\mathcal{F}; n, \ell) \cdot n^{1/q}$ play roughly the same role. Although the definition of $\mathcal{B}(\mathcal{F}; n, \ell)$ is more complicated than that of $\Gamma(\mathcal{F}; n, m)$, it will be easier to work with it, since it is "tensorized" easily, as we shall see in Lemma 2.2.

LEMMA 2.1. *For every metric space* $(\mathcal{M}, d_{\mathcal{M}})$, *every* $n, a \in \mathbb{N}$, *every even* $m, r \in \mathbb{N}$ *with* $0 \le r < m$, *and every* $f : \mathbb{Z}_m^n \to \mathcal{M}$,

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}} \left(f\left(x+(am+r)e_j\right), f(x)\right)^2 d\mu(x) \le \min\left\{r^2, (m-r)^2\right\} n \cdot \mathop{\mathbb{E}}_{\varepsilon \in \{\pm 1\}^n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^2 d\mu(x). \tag{2.8}$$

In particular, $\mathcal{B}(\mathcal{M}; n, \ell) \le 1$ *for every* $n \in \mathbb{N}$ *and every even* $\ell \in \mathbb{N}$.

Proof. The left-hand side of (2.8) depends only on r, and remains unchanged if we replace r by $m-r$. We may thus assume that $a = 0$ and $r \le m-r$. Fix $x \in \mathbb{Z}_m^n$ and $j \in \{1, \dots n\}$. Observe that

$$\left\{x + \tfrac{1-(-1)^k}{2} \sum_{r \ne j} e_r + k e_j\right\}_{k=0}^{r}$$

is a path of length r joining x and $x + re_j$ in the graph $\mathbf{diag}(\mathbb{Z}_m^n)$. Thus the distance between x and $x + re_j$ in the graph $\mathbf{diag}(\mathbb{Z}_m^n)$ equals r. If $(x = w_0, w_1, \dots, w_r = x + re_j)$ is a geodesic joining x and $x + re_j$ in $\mathbf{diag}(\mathbb{Z}_m^n)$, then by the triangle inequality

$$d_{\mathcal{M}}(f(x+re_j), f(x))^2 \le r \sum_{k=1}^{r} d_{\mathcal{M}}(f(w_k), f(w_{k-1}))^2. \tag{2.9}$$

Observe that if we sum (2.9) over all geodesics joining x and $x + re_j$ in $\mathbf{diag}(\mathbb{Z}_m^n)$, and then over all $x \in \mathbb{Z}_m^n$, then in the resulting sum each edge in $\mathbf{diag}(\mathbb{Z}_m^n)$ appears the same number of times. Thus, averaging this inequality over $x \in \mathbb{Z}_m^n$ we get that

$$\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+re_j), f(x))^2 d\mu(x) \le r^2 \mathop{\mathbb{E}}_{\varepsilon \in \{\pm 1\}^n} [d_{\mathcal{M}}(f(x+\varepsilon), f(x))]^2.$$

Summing over $j = 1, \dots, n$ we obtain the assertion. □

Sub-multiplicativity is a key property of the $\mathcal{B}(\mathcal{F}; \ell, s)$.

LEMMA 2.2. *For every four integers* $\ell, k, s, t \in \mathbb{N}$, $\mathcal{B}(\mathcal{F}; \ell k, st) \le \mathcal{B}(\mathcal{F}; \ell, s) \cdot \mathcal{B}(\mathcal{F}; k, t)$.

Proof. Let m be an even integer and take a function $f : \mathbb{Z}_m^{\ell k} \to \mathcal{M}$, $\mathcal{M} \in \mathcal{F}$. Fix $x \in \mathbb{Z}_m^{\ell k}$ and $\varepsilon \in \{-1, 1\}^{\ell k}$. Define $g : \mathbb{Z}_m^{\ell} \to \mathcal{M}$ by $g(y) = f\big(x + \sum_{r=1}^{k} \sum_{j=1}^{\ell} \varepsilon_{j+(r-1)\ell} \cdot y_j \cdot e_{j+(r-1)\ell}\big)$. By the definition of $\mathcal{B}(\mathcal{F}; \ell, s)$, applied to g, for every $\mathcal{B}_1 > \mathcal{B}(\mathcal{F}; \ell, s)$ we have that

$$\begin{aligned}
&\sum_{a=1}^{\ell} \int_{\mathbb{Z}_m^{\ell}} d_{\mathcal{M}}\Big(f\big(x + \sum_{r=1}^{k}\sum_{j=1}^{\ell} \varepsilon_{j+(r-1)\ell} \cdot y_j \cdot e_{j+(r-1)\ell} + s \sum_{r=1}^{k} \varepsilon_{a+(r-1)\ell} \cdot e_{a+(r-1)\ell}\big), \\
&\qquad f\big(x + \sum_{r=1}^{k}\sum_{j=1}^{\ell} \varepsilon_{j+(r-1)\ell} \cdot y_j \cdot e_{j+(r-1)\ell}\big)\Big)^2 d\mu_{\mathbb{Z}_m^{\ell}}(y) \\
&\le \mathcal{B}_1^2 s^2 \ell \cdot \mathop{\mathbb{E}}_{\delta \in \{\pm 1\}^{\ell}} \int_{\mathbb{Z}_m^{\ell}} d_{\mathcal{M}}\Big(f\big(x + \sum_{r=1}^{k}\sum_{j=1}^{\ell} \varepsilon_{j+(r-1)\ell} \cdot (y_j + \delta_j) \cdot e_{j+(r-1)\ell}\big), \\
&\qquad f\big(x + \sum_{r=1}^{k}\sum_{j=1}^{\ell} \varepsilon_{j+(r-1)\ell} \cdot y_j \cdot e_{j+(r-1)\ell}\big)\Big)^2 d\mu_{\mathbb{Z}_m^{\ell}}(y).
\end{aligned}$$

Averaging this inequality over $x \in \mathbb{Z}_m^{\ell k}$ and $\varepsilon \in \{-1, 1\}^{\ell k}$, and using the translation invariance of the Haar measure, we get that

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\varepsilon \in \{\pm 1\}^{\ell k}} \sum_{a=1}^{\ell} \int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\Big(f\big(x + s\sum_{r=1}^{k} \varepsilon_{a+(r-1)\ell} \cdot e_{a+(r-1)\ell}\big), f(x)\Big)^2 d\mu_{\mathbb{Z}_m^{\ell k}}(x) \\
&\le \mathcal{B}_1^2 s^2 \ell \mathop{\mathbb{E}}_{\varepsilon \in \{\pm 1\}^{\ell k}} \int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\left(f\left(x+\varepsilon\right), f\left(x\right)\right)^2 d\mu_{\mathbb{Z}_m^{\ell k}}(x).
\end{aligned} \tag{2.10}$$

Next we fix $x \in \mathbb{Z}_m^{\ell k}$, $u \in \{1, \dots, \ell\}$, and define $h_u : \mathbb{Z}_m^k \to \mathcal{M}$ by $h_u(y) = f\big(x + s\sum_{r=1}^{k} y_r \cdot e_{u+(r-1)\ell}\big)$. By the definition of $\mathcal{B}(\mathcal{F}; k, t)$, applied to h_u, for every $\mathcal{B}_2 > \mathcal{B}(\mathcal{F}; k, t)$ we have that

$$\begin{aligned}
&\sum_{j=1}^{k} \int_{\mathbb{Z}_m^{k}} d_{\mathcal{M}}\Big(f\big(x + s\sum_{r=1}^{k} y_r \cdot e_{u+(r-1)\ell} + st \cdot e_{u+(j-1)\ell}\big), \\
&\qquad f\big(x + s\sum_{r=1}^{k} y_r \cdot e_{u+(r-1)\ell}\big)\Big)^2 d\mu_{\mathbb{Z}_m^{k}}(y) \\
&= \sum_{j=1}^{k} \int_{\mathbb{Z}_m^{k}} d_{\mathcal{M}}\big(h_u(y + te_j), h_u(y)\big)^2 d\mu_{\mathbb{Z}_m^{k}}(y)
\end{aligned}$$

$$\leq \mathcal{B}_2^2 t^2 k \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^{\ell k}} \int_{\mathbb{Z}_m^k} d_M\big(h_u(y+\varepsilon), h_u(y)\big)^2 d\mu_{\mathbb{Z}_m^k}(y)$$
$$= \mathcal{B}_2^2 t^2 k \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^{\ell k}} \int_{\mathbb{Z}_m^k} d_{\mathcal{M}}\Big(f\big(x+s\sum_{r=1}^k (y_r+\varepsilon_{u+(r-1)\ell})\cdot e_{u+(r-1)\ell}\big), f\big(x+s\sum_{r=1}^k y_r\cdot e_{u+(r-1)\ell}\big)\Big)^2 d\mu_{\mathbb{Z}_m^k}(y).$$

Summing this inequality over $u\in\{1,\dots,\ell\}$ and averaging over $x\in\mathbb{Z}_m^{\ell k}$, we get, using (2.10), that

$$\sum_{a=1}^{\ell k}\int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\left(f\left(x+ste_a\right), f(x)\right)^2 d\mu(x)$$
$$\leq \mathcal{B}_2^2 t^2 k \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^{\ell k}} \sum_{u=1}^{\ell}\int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\Big(f\big(x+s\sum_{r=1}^k \varepsilon_{u+(r-1)\ell}\cdot e_{u+(r-1)\ell}\big), f\left(x\right)\Big)^2 d\mu(x)$$
$$\leq \mathcal{B}_2^2 t^2 k \mathcal{B}_1^2 s^2 \ell \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^{\ell k}} \int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\left(f\left(x+\varepsilon\right), f\left(x\right)\right)^2 d\mu(x).$$

This implies the required result. □

LEMMA 2.3. *Assume that there exist integers $n_0,\ell_0>1$ such that $\mathcal{B}(\mathcal{M};n_0,\ell_0)<1$. Then there exists $0<q<\infty$ such that for every integer n, $\Gamma_q^{(2)}(\mathcal{M})<\infty$.*

Sketch of a Proof. Let $q<\infty$ satisfy $\mathcal{B}(\mathcal{M},n_0,\ell_0)<n_0^{-1/q}$. Iterating Lemma 2.2 we get that for every integer k, $\mathcal{B}(n_0^k,\ell_0^k)\leq n_0^{-k/q}$. Denoting $n=n_0^k$ and $m=2\ell_0^k$, this implies that for every $f:\mathbb{Z}_m^n\to\mathcal{M}$,

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\frac{m}{2}e_j\right), f(x)\right)^2 d\mu(x) \leq \frac{1}{4}m^2 n^{1-\frac{2}{q}} \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon),f(x))^2 d\mu(x). \tag{2.11}$$

Inequality (2.11) "almost implies" that $\Gamma_q^{(2)}(\mathcal{M};n_0^k,2\ell_0^k)=O(1)$, except that the averaging of ε is done over $\{-1,1\}^n$ instead of $\{-1,0,1\}^n$. This gap is overcome by averaging (2.11) over all dimensions at most n — details are omitted. Extending the inequality to all n (and not just powers of n_0) is done by a simple interpolation argument — details are omitted. □

LEMMA 2.4. *Let $n>1$ be an integer, m an even integer, and s an integer divisible by 4. Assume that $\eta\in(0,1)$ satisfies $8^{sn}\sqrt{\eta}<\frac{1}{2}$, and that there exists a mapping $f:\mathbb{Z}_m^n\to\mathcal{M}$ such that*

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+se_j\right), f(x)\right)^2 d\mu(x) > (1-\eta)s^2 n \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon),f(x))^2 d\mu(x). \tag{2.12}$$

Then $c_{\mathcal{M}}\left([s/4]_\infty^n\right)\leq 1+8^{sn}\sqrt{\eta}$.
In particular, if $\mathcal{B}(\mathcal{M};n,s)=1$ then $c_{\mathcal{M}}\left([s/4]_\infty^n\right)=1$.

As the proof of Lemma 2.4 is too long to fit the current format, we illustrate it by proving a weaker assertion.

PROPOSITION 2.1. *Let $n>1$ be an integer, m an even integer, and s an integer divisible by 4. Assume that there exists a mapping $f:\mathbb{Z}_m^n\to\mathcal{M}$ such that*

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+se_j\right), f(x)\right)^2 d\mu(x) \geq s^2 n \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon),f(x))^2 d\mu(x). \tag{2.13}$$

Then $c_{\mathcal{M}}\left([s/4]_\infty^n\right)=1$.

Proof. Observe first of all that (2.13) and Lemma 2.1 imply that $m\geq 2s$. In what follows we will use the following numerical fact: If $a_1,\dots,a_r\geq 0$ and $0\leq b\leq\frac{1}{r}\sum_{j=1}^r a_j$, then

$$\sum_{j=1}^r (a_j-b)^2 \leq \sum_{j=1}^r a_j^2 - rb^2. \tag{2.14}$$

For $x\in\mathbb{Z}_m^n$ let $\mathcal{G}_j^+(x)$ (resp. $\mathcal{G}_j^-(x)$) be the set of all geodesics joining x and $x+se_j$ (resp. $x-se_j$) in the graph $\mathbf{diag}(\mathbb{Z}_m^n)$. As we have seen in the proof of Lemma 2.1, since s is even, these sets are nonempty. Notice that if $m=2s$ then $\mathcal{G}_j^+(x)=\mathcal{G}_j^-(x)$, otherwise $\mathcal{G}_j^+(x)\cap\mathcal{G}_j^-(x)=\emptyset$. Denote by $\mathcal{G}_j^\pm(x)=\mathcal{G}_j^+(x)\cup\mathcal{G}_j^-(x)$, and for $\pi\in\mathcal{G}_j^\pm(x)$,

$$\mathrm{sg}(\pi)=\begin{cases}+1 & \text{if } \pi\in\mathcal{G}_j^+(x)\\ -1 & \text{otherwise .}\end{cases}$$

Each geodesic in $\mathcal{G}_j^\pm(x)$ has length s. We write each $\pi\in\mathcal{G}_j^\pm(x)$ as a sequence of vertices $\pi=(\pi_0=x,\pi_1,\dots,\pi_s=x+\mathrm{sg}(\pi)se_j)$. Using (2.14) with $a_j=d_{\mathcal{M}}(f(\pi_j),f(\pi_{j-1}))$ and $b=\frac{1}{s}d_{\mathcal{M}}\left(f\left(x+se_j\right),f(x)\right)$, which satisfy the the conditions of (2.14) due to the triangle inequality, we get that for each $\pi \in \mathcal{G}_j^{\pm}(x)$,

$$\begin{aligned}(2.15)\quad \sum_{\ell=1}^{s} \big[d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1})) \\ - \tfrac{1}{s} d_{\mathcal{M}}\left(f\left(x+\mathrm{sg}(\pi) s e_j\right), f(x)\right)\big]^2 \\ \le \sum_{k=1}^{s} d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1}))^2 \\ - \tfrac{1}{s} d_{\mathcal{M}}\left(f\left(x+\mathrm{sg}(\pi) s e_j\right), f(x)\right)^2 .\end{aligned}$$

By symmetry $|\mathcal{G}_j^+(x)| = |\mathcal{G}_j^-(x)|$, and this value is independent of $x \in \mathbb{Z}_m^n$ and $j \in \{1,\dots,n\}$. Denote $g = |\mathcal{G}_j^{\pm}(x)|$, and observe that $g \le 2 \cdot 2^{ns}$. Averaging (2.15) over all $x \in \mathbb{Z}_m^n$ and $\pi \in \mathcal{G}_j^{\pm}(x)$, and summing over $j \in \{1,\dots,n\}$, we get that

$$\begin{aligned}\frac{1}{g}\sum_{j=1}^{n}\int_{\mathbb{Z}_m^n}\sum_{\pi\in\mathcal{G}_j^{\pm}(x)}\sum_{\ell=1}^{s}\big[d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1})) \\ - \tfrac{1}{s} d_{\mathcal{M}}\left(f\left(x+\mathrm{sg}(\pi) s e_j\right), f(x)\right)\big]^2 d\mu(x) \\ \le sn\,\mathbb{E}_{\varepsilon}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^2\, d\mu(x) \\ - \frac{1}{s}\sum_{j=1}^{n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+se_j\right), f(x)\right)^2\, d\mu(x) \\ (2.16)\quad \le 0.\end{aligned}$$

Thus, (2.16) implies that for every $x \in \mathbb{Z}_m^n$, every $j \in \{1,\dots,n\}$, every $\pi \in \mathcal{G}_j^{\pm}(x)$, and every $\ell \in \{1,\dots,s\}$,

$$(2.17)\quad d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1})) = \frac{1}{s} d_{\mathcal{M}}\left(f\left(x+\mathrm{sg}(\pi) s e_j\right), f(y)\right)$$

CLAIM 2.1. *For every $x \in \mathbb{Z}_m^n$, and $\varepsilon, \delta \in \{-1,1\}^n$,*

$$d_{\mathcal{M}}(f(x), f(x+\varepsilon)) = d_{\mathcal{M}}(f(x), f(x+\delta)).$$

Sketch of a Proof. If $\delta = \varepsilon$ there is nothing to proves. Otherwise the two pairs $(x+\delta, x)$, and $(x, x+\varepsilon)$ are clearly part of some geodesic in $\mathcal{G}_j^{\pm}(x+\delta)$, for some $j \in \{1,\dots,n\}$, and (2.17) implies their equality. □

CLAIM 2.2. *For every $x, y \in \mathbb{Z}_m^n$, and $\varepsilon, \delta \in \{-1,1\}^n$, $d_{\mathcal{M}}(f(x), f(x+\varepsilon)) = d_{\mathcal{M}}(f(y), f(y+\delta))$.*

Proof. Take any path in $\mathbf{diag}(\mathbb{Z}_m^n)$ containing both $(x, x+\varepsilon)$, and $(y, y+\delta)$ and apply Claim 2.1 for every consecutive pair of edges along this path. □

Without loss of generality we scale the distances to satisfy $d_{\mathcal{M}}(f(x), f(x+\varepsilon)) = 1$, for any $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1,1\}^n$.

CLAIM 2.3. *Denote $V = \big\{x \in \mathbb{Z}_m^n : \forall j\ 0 \le x_j \le \frac{s}{2}$ and x_j is even$\big\}$. Then the following assertions hold true:*

1. *For every $x, y \in V$ there exists $j \in \{1,\dots,n\}$, and a path $\pi \in \mathcal{G}_j^+(x)$ of length s which goes through x and y.*
2. *For every $x, y \in V$, $d_{\mathbf{diag}(\mathbb{Z}_m^n)}(x,y) = d_{\mathbb{Z}_m^n}(x,y) = \|x-y\|_\infty$.*

Sketch of a Proof. Let $j \in \{1,\dots,n\}$ be such that $|y_j - x_j| = \|x-y\|_\infty$. We can take a geodesic π' connecting x with y, and then concatenating π' with π'', where in π'' the edges are reversed relative to π, except in the j-th coordinate. Thus, $\pi' \circ \pi''$ connect x, y, and $x + 2\|x-y\|_\infty e_j$. The path can now easily be continued to $x + se_j$ in $\mathbf{diag}(\mathbb{Z}_m^n)$, since s is even.

The second assertion is obvious. □

COROLLARY 2.1. $\forall x, y \in V$, $d_{\mathcal{M}}(f(x), f(y)) = \|x-y\|_\infty$.

Proof. Equality (2.17) implies that the $d_{\mathcal{M}}(f(x), f(x+se_j)) = s$, so applying the triangle inequality twice on the path constructed in Claim 2.3 concludes the proof. □

This concludes the proof of Proposition 2.1, since the mapping $x \mapsto x/2$ is a distortion 1 bijection between $(V, d_{\mathbb{Z}_m^n})$ and $[s/4]_\infty^n$. □

LEMMA 2.5. *Let $\mathcal{F}$ be a a family of metric spaces. Fix $q < \infty$ and assume that $\Gamma_q^{(2)}(\mathcal{F}; n, m) < \infty$. Then $c_{\mathcal{F}}\big(\mathbb{Z}_m^n\big) \ge n^{1/q}/(2\Gamma_q^{(2)}(\mathcal{F}; n, m))$.*

Proof. Fix $\mathcal{M} \in \mathcal{F}$, a bijection $f : \mathbb{Z}_m^n \to \mathcal{M}$, and $\Gamma > \Gamma_q^{(2)}(\mathcal{F}; n, m)$. Then

$$\begin{aligned}\frac{nm^2}{4\|f^{-1}\|_{\mathrm{Lip}}^2} &\le \sum_{j=1}^{n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\frac{m}{2}e_j\right), f(x)\right)^2 d\mu(x) \\ &\le \Gamma^2 m^2 n^{1-\frac{2}{q}} \\ &\quad \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1,0\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^2 d\mu(x) \\ &\le \Gamma^2 m^2 n^{1-\frac{2}{q}}\|f\|_{\mathrm{Lip}}^2.\end{aligned}$$

It follows that $\mathrm{dist}(f) \ge n^{1/q}/2\Gamma$. □

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. We first assume that $\Gamma_q^{(2)}(\mathcal{F}) = \infty$ for all $q < \infty$. By Lemma 2.3 it follows that for every two integers $n, s > 1$, $\mathcal{B}(\mathcal{F}; n, s) = 1$. Now the required result follows from Lemma 2.4.

In the other direction, assume $\Gamma_q^{(2)}(\mathcal{F}) < \infty$. By Lemma 2.5, for any n there exists m such that $c_{\mathcal{F}}\big(\mathbb{Z}_m^n\big) \ge n^{1/q}/(2\Gamma_q^{(2)}(\mathcal{F}))$. Since $c_{[m]_\infty^{mn}}(\mathbb{Z}_m^n) = 1$, this implies that there exist $m, n \in \mathbb{N}$, for which $c_{\mathcal{F}}([m]_\infty^n) > 1$. □

3 Relative Conductance

In this section we discuss the proof of Theorem 1.5,

We begin with the lower bound on $\Phi_{E_2/E_1}(\mathbb{Z}_m^n)$. Consider a cut $(S, \bar{S})$ of $\mathbb{Z}_m^n$. We can associate with the cut a mapping $f : \mathbb{Z}_m^n \to \{0, 1\}$,

$$f(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{otherwise.} \end{cases}$$

Fix $\Gamma > \Gamma_2^{(1)}(L_1)$. Since $\{0, 1\}$ is an L_1 metric, we can apply (1.5), and obtain that $\forall n \in \mathbb{N}$, $\exists m \in 2\mathbb{N}$,

$$\begin{aligned} m^{-n}|E_1(S, \bar{S})| &= \sum_{j=1}^{n} \mathop{\mathbb{E}}_{x\in\mathbb{Z}_m^n} \Big[d_{\mathcal{M}}\big(f(x + \tfrac{m}{2}e_j), f(x)\big)\Big] \\ &\le \Gamma m n^{1/2} \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1,0\}^n} \mathop{\mathbb{E}}_{x\in\mathbb{Z}_m^n} [d_{\mathcal{M}}(f(x + \varepsilon), f(x))] \\ &= \Gamma m n^{1/2}(3m)^{-n}|E_2(S, \bar{S})|. \end{aligned}$$

This implies that for every $n \in \mathbb{N}$, there exists $m \in 2\mathbb{N}$,

$$\Phi_{E_2/E_1}(\mathbb{Z}_m^n) = \frac{n|E_2(S, \bar{S})|}{3^n|E_1(S, \bar{S})|} \ge \Gamma^{-1} \frac{\sqrt{n}}{m}. \tag{3.18}$$

Note that Def. 1.2 only guarentees the *existence* of m for which (3.18) holds. However, in the full version we also investigate what is the value of m for which (1.5) holds, and in this case we actually have that $\forall n \in \mathbb{N}$, $\forall t \in 4\mathbb{N}$, set $m = t\lceil\sqrt{n}\rceil$. Then $\forall f : \mathbb{Z}_m^n \to L_1$,

$$\begin{aligned} &\sum_{j=1}^{n} \mathop{\mathbb{E}}_{x\in\mathbb{Z}_m^n} \Big[d_{\mathcal{M}}\big(f(x + \tfrac{m}{2}e_j), f(x)\big)\Big] \\ &\qquad \le \Gamma m n^{1/2} \mathop{\mathbb{E}}_{\varepsilon\in\{\pm1,0\}^n} \mathop{\mathbb{E}}_{x\in\mathbb{Z}_m^n} [d_{\mathcal{M}}(f(x + \varepsilon), f(x))], \end{aligned}$$

This proves that $\Phi_{E_2/E_1}(\mathbb{Z}^n_{t\sqrt{n}}) \ge \Omega(1/t)$. The extension to any m divisible by 4, follows from more elobaration of these techniques.

The upper bound in Theorem 1.5 follows from embedding of $\mathbb{Z}_m^n$ in L_1, as we now explain.

LEMMA 3.1. $\forall n \in \mathbb{N}$ $\forall m \in 2\mathbb{N}$, $c_{L_1}(\mathbb{Z}_m^n) \ge \frac{m}{2} \cdot \Phi_{E_2/E_1}(Z_m^n)$.

Proof. Consider an embedding $f : \mathbb{Z}_m^n \to L_1$. Since finite L_1 metrics are in the cut cone, there exists $\alpha_S \ge 0$, for $\emptyset \ne S \subseteq \mathbb{Z}_m^n$ such that for any $x, y \in \mathbb{Z}_m^n$, $\|f(x) - f(y)\|_1 = \sum_{S\subset\mathbb{Z}_m^n} \alpha_S\delta_S(x, y)$, where $\delta_S(x, y)$ is 1 when $|S \cap \{x, y\}| = 1$ and 0 otherwise. Then

$$\begin{aligned} &\|f\|_{\text{Lip}} \cdot \|f^{-1}\|_{\text{Lip}} \cdot \frac{2}{m} \\ &= \|f\|_{\text{Lip}}\|f^{-1}\|_{\text{Lip}} \frac{|E_1| \cdot \sum_{(x,y)\in E_2} d_{\mathbb{Z}_m^n}(x, y)}{|E_2| \cdot \sum_{(x,y)\in E_1} d_{\mathbb{Z}_m^n}(x, y)} \\ &\ge \frac{|E_1| \cdot \sum_{(x,y)\in E_2} \|f(x) - f(y)\|_1}{|E_2| \cdot \sum_{(x,y)\in E_1} \|f(x) - f(y)\|_1} \\ &= \frac{|E_1| \cdot \sum_{\emptyset\ne S\subseteq\mathbb{Z}_m^n} \alpha_S \sum_{(x,y)\in E_2} \delta_S(x, y)}{|E_2| \cdot \sum_{\emptyset\ne S\subseteq\mathbb{Z}_m^n} \alpha_S \sum_{(x,y)\in E_1} \delta_S(x, y)} \\ &\ge \min_{\emptyset\ne S\subseteq\mathbb{Z}_m^n} \frac{|E_1| \cdot |E_2(S, \bar{S})|}{|E_2| \cdot |E_1(S, \bar{S})|} \\ &= \Phi_{E_2/E_1}(\mathbb{Z}_m^n). \end{aligned}$$

□

On the other hand $c_{L_1}(\mathbb{Z}_m^n) \le c_{L_2}(\mathbb{Z}_m^n) \le \min\{\frac{m}{2}, \frac{\pi}{2}\sqrt{n}\}$, which implies that $\Phi_{E_2/E_1}(\mathbb{Z}_m^n) \le \min\{1, \pi\frac{\sqrt{n}}{m}\}$.

4 Coarse Embedding

In this section we prove Theorem 1.6.

Let $(\mathcal{N}, d_{\mathcal{N}})$ and $(\mathcal{M}, d_{\mathcal{M}})$ be metric spaces. For $f : \mathcal{N} \to \mathcal{M}$ and $t > 0$ we define

$$\Omega_f(t) = \sup\{d_{\mathcal{M}}(f(x), f(y));\ d_{\mathcal{N}}(x, y) \le t\}, \quad \text{and}$$
$$\omega_f(t) = \inf\{d_{\mathcal{M}}(f(x), f(y));\ d_{\mathcal{N}}(x, y) \ge t\}.$$

Clearly Ω_f and ω_f are non-decreasing, and for every $x, y \in \mathcal{N}$, $\omega_f(d_{\mathcal{N}}(x, y)) \le d_{\mathcal{M}}(f(x), f(y)) \le \Omega_f(d_{\mathcal{N}}(x, y))$. With these definitions, f is a coarse embedding if $\Omega_f(t) < \infty$ for all $t > 0$ and $\lim_{t\to\infty} \omega_f(t) = \infty$.

DEFINITION 4.1. *Denote by $m_q^{(p)}(\mathcal{M}; n, \Gamma)$ the smallest even integer m for which* (1.5) *holds. As before, when $p = q$ we write $m_q^{(p)}(\mathcal{M}; n, \Gamma) = m_q(\mathcal{M}; n, \Gamma)$.*

LEMMA 4.1. *Let $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space which contains at least two points. Then for every integer n, every $\Gamma > 0$, and every $q > 0$, $m_q(\mathcal{M}; n, \Gamma) \ge n^{1/q}/\Gamma$.*

Proof. Fix $u, v \in \mathcal{M}$, and without loss of generality normalize the metric so that $d_{\mathcal{M}}(u, v) = 1$. Denote $m = m_q(\mathcal{M}; n, \Gamma)$. Let $f : \mathbb{Z}_m^n \to \mathcal{M}$ be the random mapping such that for every $x \in \mathbb{Z}_m^n$, $\Pr[f(x) = u] = \Pr[f(x) = v] = \frac{1}{2}$, and $\{f(x)\}_{x\in\mathbb{Z}_m^n}$ are independent random variables. Then for every distinct $x, y \in \mathbb{Z}_m^n$, $\mathbb{E}[d_{\mathcal{M}}(f(x), f(y))^p] = \frac{1}{2}$. Thus the required result follows by applying (2.6) to f, and taking expectation. □

LEMMA 4.2. *Let $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space, n an integer, $\Gamma > 0$, and $0 < q \le r$. Then for every function $f : \ell_r^n \to \mathcal{M}$, and every $s > 0$,*

$$n^{1/q}\omega_f(2s) \le \Gamma m_q(\mathcal{M}; n, \Gamma) \cdot \Omega_f\Big(\frac{2\pi s n^{1/r}}{m_q(\mathcal{M}; n, \Gamma)}\Big).$$

Proof. Denote $m = m_q^{(p)}(\mathcal{M}; n, \Gamma)$, and define $g : \mathbb{Z}_m^n \to \mathcal{M}$ by

$$g(x_1, \ldots, x_n) = f\Big(\sum_{j=1}^{n} se^{\frac{2\pi i x_j}{m}} e_j\Big).$$

Then

$$\begin{aligned}
&\int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(g(x+\varepsilon), g(x))^q d\mu(x) d\sigma(\varepsilon) \\
\le\ & \max_{\varepsilon \in \{-1,01,\}^n} \Omega_f\left(s\left(\sum_{j=1}^{n}\left|e^{\frac{2\pi i \varepsilon_j}{m}} - 1\right|^r\right)^{1/r}\right)^q \\
\le\ & \Omega_f\Big(\frac{2\pi s n^{1/r}}{m}\Big)^q.
\end{aligned}$$

On the other hand,

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x + \tfrac{m}{2} e_j\right), f(x)\right)^q d\mu(x) \ge n\omega_f(2s)^q.$$

By the definition of $m_q(\mathcal{M}; n, \Gamma)$ it follows that

$$n\omega_f(2s)^q \le \Gamma^q m^q \Omega_f\Big(\frac{2\pi s n^{1/r}}{m}\Big)^q,$$

as required. □

COROLLARY 4.1. *Let $\mathcal{M}$ be a metric space and assume that there exist constants $c, \Gamma > 0$ such that for infinitely many integers n, $m_q(\mathcal{M}; n, \Gamma) \le cn^{1/q}$. Then for every $r > q$, ℓ_r does not coarsely embed into $\mathcal{M}$.*

Proof. Choose $s = n^{\frac{1}{q} - \frac{1}{r}}$ in Lemma 4.2. Using Lemma 4.1 we get that $\omega_f\big(2n^{\frac{1}{q} - \frac{1}{r}}\big) \le c\Gamma\Omega_f\left(2\pi\Gamma\right)$. Since $q < r$, it follows that $\liminf_{t\to\infty} \omega_f(t) < \infty$, so f is not a coarse embedding. □

In the full version of this paper we prove the following strengthening of Theorem 1.3 for Banach spaces with type larger than 1.

THEOREM 4.1. *Let X be a Banach space with type larger than 1 and cotype q. Then there exists $\Gamma > 0$ for which $m_q(X; n, \Gamma) = O(n^{1/q})$.*

Proof of Theorem 1.6. Assume first that $q \ge 2$. then L_q has type 2, and by Theorem 4.1, there exists $\Gamma > 0$ for which $m_q(L_q; n, \Gamma) = O(n^{1/q})$. By Corollary 4.1, ℓ_r does not coarsely embed in L_q. When $q \in [1, 2)$, we use the well known fact that L_q coarsely embeds in L_2 [37]. □

Acknowledgments We are grateful to Keith Ball for several valuable discussions. We thank Yuri Rabinovich for pointing out the connection to Matoušek's BD Ramsey theorem. Comments from the SODA's referees helped in improving the presentation.

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