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A Note on Bipartite Graphs without $2k$-Cycles.

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The question of the maximum number $\mathrm{ex}(m,n,C_{2k})$ of edges in an m by n bipartite graph without a cycle of length $2k$ is addressed in this note. For each $k \geq 2$, it is shown that

$$\mathrm{ex}(m,n,C_{2k}) \leq \begin{cases} (2k-3)\left[(mn)^{\frac{k+1}{2k}} + m + n\right] & \text{if } k \text{ is odd.} \\ (2k-3)\left[m^{\frac{k+2}{2k}} n^{\frac{1}{2}} + m + n\right] & \text{if } k \text{ is even.} \end{cases}$$

1. Introduction

In this note, we study the maximum number of edges in an m by n bipartite graph containing no $2k$-cycles. This problem was studied in the papers by Erdős, Sós and Sárközy [3] and by Győri [4], in the context of a number theoretic problem. It is also related to the size of subsets of points in projective planes, such as arcs and caps. A connection to a geometric problem involving points and lines in Euclidean space is described in de Caen and Székely [1].

Throughout the material to follow, $\gamma(m,n,g,k)$ denotes the maximum number of edges in an m by n bipartite graph of girth at least $2g$ containing no $2k$-cycle. In particular, we write $\gamma(m,n,2,k) = \mathrm{ex}(m,n,C_{2k})$, which is the maximum number of edges in a $2k$-cycle-free m by n bipartite graph. For $d,g \geq 2$, let $c(d,g)$ be the largest integer such that every bipartite graph of average degree at least $2d$ and girth at least $2g$ contains a cycle of length at least $c(d,g)$ with at least one chord. Our main result is as follows:

Theorem 1. *Let $k,g \geq 2$ be integers. Then, for any d such that $c(d,g) \geq 2(k-g+1)$,*

$$\gamma(m,n,g,k) \leq \begin{cases} 2d\left[(mn)^{\frac{k+1}{2k}} + m + n\right] & \text{if } k \text{ is odd.} \\ 2d\left[m^{\frac{k+2}{2k}} n^{\frac{1}{2}} + m + n\right] & \text{if } k \text{ is even.} \end{cases}$$

Theorem 1 depends explicity on the parameter $c(d,g)$. Let us make a few remarks about

$c(d,g)$. First, a straightforward argument shows that $c(d,g) \geq 2d(g-1)+1$. Stronger results were obtained by Erdős, Faudree, Rousseau, and Schelp [2]. We deduce from their paper that $c(d,g) \geq d^{g/4}$. In particular, by using these two bounds on $c(d,g)$ in Theorem 1, we respectively obtain the following two corollaries:

Corollary 2.

$$ex(m,n,C_{2k}) \quad \leq \quad \begin{cases} (2k-3)\Big[(mn)^{\frac{k+1}{2k}} + m + n\Big] & \text{if } k \text{ is odd.} \\ (2k-3)\Big[m^{\frac{k+2}{2k}}\, n^{\frac{1}{2}} + m + n\Big] & \text{if } k \text{ is even.} \end{cases}$$

Corollary 3. *For any number $\delta > 0$, there exists a constant $c(\delta)$ such that*

$$\gamma(m,n,\delta\log k,k) \quad \leq \quad \begin{cases} c(\delta)\Big[(mn)^{\frac{k+1}{2k}} + m + n\Big] & \text{if } k \text{ is odd.} \\ c(\delta)\Big[m^{\frac{k+2}{2k}}\, n^{\frac{1}{2}} + m + n\Big] & \text{if } k \text{ is even.} \end{cases}$$

Using the requirement on $c(d,g)$ from Theorem 1 and the inequality $c(d,g) \geq d^{g/4}$, a short computation shows that we may certainly take $c(\delta) = 4^{4/\delta}$ in Corollary 3. Recent results of Hoory [5] and of Lam [7] show that $\gamma(m,n,k,k)$ satisfies the bounds given in Corollary 3, with $c(\delta) = 1$. In the case $k \in \{2,3,5\}$, the existence of rank two geometries known as generalized polygons (see [6] for constructions) show that the constant $c(\delta) = 1$ is best possible when $m = n$. The strength in Corollary 3 is that it shows that excluding cycles of length $O(\log k)$ has, to within an absolute constant factor, the same effect on the upper bounds as excluding all cycles of length at most $2k$. On the other hand, our next result gives an indication that $\gamma(m,n,k,k)$ and $\gamma(m,n,2,k)$ may differ substantially:

Theorem 4. *For all integers m,n and $k \geq 3$,*

$$\gamma((k-1)m,n,k,k) \quad \geq \quad (k-1)\cdot\gamma(m,n,2,k).$$

If we make the assumption that for each even positive integer k, $\gamma(m,n,2,k)$ is asymptotically $c_1(mn)^{1/2+1/(2k)} + c_2(m+n)$ as $m,n \to \infty$, for some constants c_1, c_2, then Theorem 4 gives

$$\liminf_{m,n\to\infty} \frac{\gamma(m,n,k,k)}{\gamma(m,n,2,k)} \quad \geq \quad \sqrt{k-1}.$$

Similar observations may be made for odd values of k, namely

$$\liminf_{m,n\to\infty} \frac{\gamma(m,n,k,k)}{\gamma(m,n,2,k)} \quad \geq \quad (k-1)^{1/2-1/(2k)}.$$

One can deduce from these observations and the constructions of generalized quadrangles and hexagons that

$$\liminf_{n\to\infty} \frac{\gamma(n,2n,3,3)}{\gamma(n,2n,2,3)} \quad \geq \quad 2^{1/3}$$

$$\liminf_{n\to\infty} \frac{\gamma(n,2n,5,5)}{\gamma(n,2n,2,5)} \quad \geq \quad 4^{1/5}$$

The first inequality comes from [8] and the second inequality is implicit in the work of Lazebnik, Ustimenko and Woldar [6]. The next section is devoted to proving Theorem 1, and the construction for Theorem 4 is presented in Section 3.

2. Proof of Theorem 1

The main tool in our proof will be the first theorem in [9]. Although the next proposition is not the statement of this theorem, it is straightforward to verify, from the proof appearing in [9]:

Proposition 5. *Let G be a bipartite graph of average degree at least $4d$ and girth $2g$. Then G contains cycles of $\frac{1}{2}c(2d, 2g)$ consecutive even lengths, the shortest of which has length at most twice the radius of G.*

Proof of Theorem 1 We proceed by induction on $m+n$. Suppose, for a contradiction, that G is a bipartite graph without cycles of length at most $2g-2$ and containing no $2k$-cycle, with parts A and B of sizes m and n, respectively, and with more edges than the corresponding upper bound in Theorem 1. By Proposition 5, and using the definition of d in the statement of Theorem 1, it is sufficient to show that G contains a subgraph of radius at most k and average degree at least d. Indeed, since G has even girth $2g$, Proposition 5 would then imply that there is an integer r such that $g \le r \le k$ and such that G contains the cycles $C_{2r}, C_{2r+2}, \ldots, C_{2r+2k-2g}$. Since $2k \in [2r, 2r+2k-2g]$ whenever $g \le r \le k$, one of these cycles has length $2k$, as required.

Case 1. k is odd.
In this case, we may assume that the minimum degree in A is at least

$$d_A = d\frac{m^{\frac{1}{2}+\frac{1}{2k}}}{n^{\frac{1}{2}-\frac{1}{2k}}} + d.$$

Indeed, if there was a vertex $v \in A$ with degree less than d_A then by deleting it we would arrive at an $m \times (n-1)$ bipartite graph G' with

$$\begin{aligned} e(G') \;>\; e(G) - d_A \;&\ge\; 2dm^{\frac{1}{2}+\frac{1}{2k}}\left(n^{\frac{1}{2}+\frac{1}{2k}} - \tfrac{1}{2}n^{-\frac{1}{2}+\frac{1}{2k}}\right) + 2d(m+n) - d \\ &>\; 2d\left([m(n-1)]^{\frac{1}{2}+\frac{1}{2k}} + (m+n-1)\right), \end{aligned}$$

so that G' contains a $2k$-gon by the inductive hypothesis. Similarly, we may assume that the minimum degree in B is at least

$$d_B = d\frac{n^{\frac{1}{2}+\frac{1}{2k}}}{m^{\frac{1}{2}-\frac{1}{2k}}} + d.$$

Choose a vertex $v \in A$ and let H_r be the subgraph of G induced by vertices at distance at most r from v. Let us show that H_r has average degree at least d for some $r \le k$. Suppose this cannot be done. Let D_r denote the set of vertices of H_r at distance exactly r from v. Then the average number of neighbors in D_{r-1} of a vertex in D_r is less than d. It follows that if $D_r \subset A$, then $|D_{r-1}|\,(d_B - d) < d|D_r|$ and if $D_r \subset B$, then $|D_{r-1}|\,(d_A - d) < d|D_r|$. Therefore,

$$|D_r| \;>\; \begin{cases} \left(\frac{d_A}{d} - 1\right)|D_{r-1}| \;=\; \frac{m^{\frac{1}{2}+\frac{1}{2k}}}{n^{\frac{1}{2}-\frac{1}{2k}}}|D_{r-1}| & \text{if } r \text{ is odd.} \\ \left(\frac{d_B}{d} - 1\right)|D_{r-1}| \;=\; \frac{n^{\frac{1}{2}+\frac{1}{2k}}}{m^{\frac{1}{2}-\frac{1}{2k}}}|D_{r-1}| & \text{if } r \text{ is even.} \end{cases}$$

Iterating these inequalities for $r = 1, 2, \ldots, k$ we get that since k is odd,

$$|D_k| \;>\; \left(\frac{m^{\frac{1}{2}+\frac{1}{2k}}}{n^{\frac{1}{2}-\frac{1}{2k}}}\right)^{\lceil \frac{k}{2} \rceil} \left(\frac{n^{\frac{1}{2}+\frac{1}{2k}}}{m^{\frac{1}{2}-\frac{1}{2k}}}\right)^{\lfloor \frac{k}{2} \rfloor} \;=\; m.$$

On the other hand, using the fact that k is odd once more, $D_k \subset B$, so that $|D_k| \leq m$, which is a contradiction. The proof of the second part is complete.

Case 2. k is even

The proof here is similar, so we only indicate the necessary changes to the argument. The inductive hypothesis implies that the minimum degree in A is at least:

$$d'_A = d\frac{m^{\frac{1}{2}+\frac{1}{k}}}{\sqrt{n}} + d,$$

and the minimum degree in B is at least:

$$d_B = d\frac{\sqrt{n}}{m^{\frac{1}{2}-\frac{1}{k}}} + d.$$

We now start with a vertex $v \in B$, and repeat the above argument. Since k is even, $D_k \subset B$, so that $|D_k| \leq m$, but

$$|D_k| > \left(\frac{m^{\frac{1}{2}+\frac{1}{k}}}{\sqrt{n}}\right)^{\frac{k}{2}} \left(\frac{\sqrt{n}}{m^{\frac{1}{2}-\frac{1}{k}}}\right)^{\frac{k}{2}} = m,$$

so we once more arrive at a contradiction. ∎

3. Proof of Theorem 4

Suppose we are given an m by n bipartite graph H, of girth at least $2k+2$. From H, we construct a $(k-1)m$ by n bipartite graph constaining no $2k$-cycles, and with $k-1$ times as many edges as H. Let A, B be the parts of H, let $A_1, A_2, \ldots, A_{k-1}$ be disjoint sets, and let $\phi : \bigcup_{i=1}^{k-1} A_i \to A$ be defined so that ϕ restricted to A_i is a bijection $A_i \leftrightarrow A$. Define a new graph G with parts $\bigcup_{i=1}^{k-1} A_i$ and B, with edge set

$$E = \{ab : \phi(a)b \in H\}.$$

In words, we are taking $(k-1)$ identical edge-disjoint copies of H which share B as one of their parts. We now show that G has no $2k$-cycles.

Suppose, for a contradiction, that G contains a $2k$-cycle $C = (a_1, b_1, a_2, a_2, \ldots, a_k, b_k, a_1)$ with $b_i \in B$ for all $i \in \{1, 2, \ldots, k\}$. Then

$$W = (\phi(a_1), b_1, \phi(a_2), b_2, \ldots, \phi(a_k), b_k, \phi(a_1))$$

is a closed walk of length $2k$ in H. As H has girth at least $2k+2$, W takes place on a tree $T \subset H$ with at most k edges. On the other hand, the tree contains the k vertices in $V(C) \cap B$, and there are at least two vertices $a, a' \in V(C) \cap A_i$ for some $i \in \{1, 2, \ldots, k-1\}$, by the pigeonhole principle. Now $\phi(a)$ and $\phi(a')$ are distinct, since ϕ restricted to A_i is a bijection. Therefore the tree has at least $k+2$ vertices, a contradiction. Therefore the graph G is $2k$-cycle-free.

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