Status Report for the Princeton University Junior Research Seminar / Undergraduate Mathematics Laboratory

August 21, 2002

1 Introduction

The purpose of the Junior Research Seminar / Undergraduate Mathematics Laboratory at Princeton is to form a research team of undergraduates, graduate students and faculty to investigate interesting unsolved conjectures theoretically and experimentally. The UML is sponsored by a VIGRE grant of the National Science Foundation.

In addition to the standard lecture-homework classes, we wanted a class where the undergraduates would work on hot conjectures and see what kinds of problems mathematicians study. In the sciences and engineering, undergraduates are often exposed to state of the art projects through experimental labs; we wanted to bring a similar experience to the math majors. We are competing with other departments (computer science, financial math) for majors; we feel interesting classes such as this help keep people, if not math majors, than at least math minors and people who appreciate and understand what we do.

The undergrads often have enough theory to understand the basic framework and proofs of simple cases. Building on this, they then numerically test the conjectures. The undergrads learn a good deal of theory, they learn about coding, simulations and optimization (very marketable skills), and they get to see what is out there. The graduate students and the faculty get a potent calculating force for numerical investigations. Many of the problems the students have investigated arise from graduate dissertations or current faculty research. It has been very easy finding graduate students and faculty excited about working on this course; at the end of a semester or year, instead of having a folder of solution keys to calculus, a graduate student should be able to co-author an experimental math paper with the undergrads.

This program is exportable; in fact, Professors Peter Sarnak and Steven Miller will also run a similar course at NYU during Fall 2002. Depending on the caliber of the students and the complexity of the problems chosen, it can be run from freshmen to seniors; obviously, the class would have a different flavor for freshmen/sophomores versus juniors/seniors. We've been running it mostly with juniors, with a few seniors.

It requires at least modest computer support, as well as an enthusiastic faculty / graduate student team. The challenge is finding interesting problems which (1) the undergrads know enough theory to understand and (2) are amenable to the undergraduates' coding abilities.

For (1), for the past two years we've arranged a parallel lecture series to the class to give a quick introduction to the needed material. For (2), we try to have an extensive list of problems of varying coding difficulty. This is not a course in programming, but a course for mathematicians using computers as a tool. Fortunately, there are numerous problems in number theory and graph theory where the mathematical prerequisites are light and the coding is straightforward.

A class like this is significantly more work than a normal class for the faculty and graduate students, but it is also a fun class to teach, and the benefits are worth the extra time.

In exporting this to other schools, it's important to remember this is not just a class for undergrads, but an opportunity to form a research group of undergrads, graduate students and faculty.

Below is a shortened version of a description of the class. For more information and to view the undergraduates' reports, please go to

$$http://www.math.princeton.edu/\sim mathlab/index.html$$

Some of the results from the Random Matrix Theory and Ramanujan Graph problems were recently presented by Steven Miller at the Computational Number Theory Workshop at the Foundations of Computational Mathematics Conference in Minneapolis (August, 2002).

For the slides of the talk, go to

$$http : //www.princeton.edu/ \sim sjmiller/math/talks/minn.ps$$

For an expanded version of the lecture, go to

$$http : //www.princeton.edu/ \sim sjmiller/math/talks/cnt\_sjm.ps$$

Steven Miller will also present a report on the Undergraduate Mathematics Lab at a National Science Foundation conference on computation (Washington, D.C., September 2002).

2 $2000 - 2001$

2.1 Class Structure

For the first third of the semester Professor Peter Sarnak and Steven Miller lectured twice a week. The first was a background talk, and the second would cover the topic in more depth and end with a list of unsolved conjectures that are amenable to experimental testing.

Topics covered were

1. Hardy-Littlewood Circle Method
2. Random Matrix Theory
3. Ramanujan Graphs
4. Elliptic Curves

By the end of the first third of the semester, each junior had chosen a problem to investigate.

During the second third of the term the juniors split their time between researching their problem and writing programs to test unsolved conjectures. Additional lectures on the above topics were now alternated with the juniors giving brief reports of their findings. Even though most students were doing very different research projects, by attending the same lectures and listening to periodic reports students learned about all the topics. Moreover, while not doing identical topics, different students often faced similar optimization

/ coding problems. Using class time to constantly discuss work resulted in brainstorming sessions between the Juniors and the faculty advisors.

In the final third of the semester the juniors finished their investigations, wrote up their reports, and gave longer presentations to the class.

2.2 Problems Investigated

The juniors investigated the following topics:

2.2.1 Random Matrix Theory

The local scaled spacing distribution between the eigenvalues of a random large symmetric matrix whose entries are chosen to be I.I.D. Gaussian is a well known celebrated result from random matrix theory. It is believed (and is a central open problem) that the same universal laws hold for the eigenvalue spacings if one chooses the matrix entries as I.I.D., but not necessarily Gaussian. There is not that much evidence for such a universality conjecture and numerical experiments would be illuminating (there are some done for matrices of size 20x20 going back to the 50's and 60's, but one can go a lot further with today's eigenvalue algorithms and computers). Two juniors investigated 400x400 matrices with elements chosen from the Poisson, Cauchy, and Uniform distributions, as well as the eigenvalue distributions of sparse and band matrices. In all cases, excellent agreement with theory was observed.

2.2.2 Ramanujan Graphs

Let X_N be a random cubic graph on N vertices, i.e., 3 edges emanate from each vertex. Let A be the adjacency matrix for X_N, i.e., $a_{vw} = 1$ if v is joined to w and 0 otherwise. A is symmetric and its eigenvalues all lie in $[-3, 3]$. The largest eigenvalue is equal to 3. Denote by $\lambda_1(X)$ the next to largest eigenvalue of A. In the construction of highly connected sparse graphs ("expanders"), it is desirable to have $\lambda_1(X)$ as small as possible. It is known that for any sequence of X_N's

$$\liminf_{N\to\infty} \lambda_1(X) \geq 2\sqrt{2} \tag{1}$$

The question is whether, with probability 1, $\lambda_1(X) \to 2\sqrt{2}$ as $N \to \infty$. If the answer to the above question is yes, then the random graph would be as good (for this purpose) as the arithmetically constructed Ramanujan graph.

Two juniors investigated large collections of random graphs with N vertices (N up to 1000). One junior investigated by generating large numbers of graphs randomly; the other by performing a random walk on the space of graphs. As the number of vertices increased, the average of the λ_1's seem to approach $2\sqrt{2}$, and the standard deviation seems to decrease.

There are known constructions for families of Ramanujan graphs using Number Theory for $(p^r + 1)$-regular graphs. The juniors also investigated 7-regular graphs (the first number with no known construction), and observed similar results.

2.2.3 Hardy-Littlewood Varieties

Let V be a variety (for example a hypersurface) defined by integral polynomial equations in $(x_1, \ldots, x_n)$. Let $V(Z)$ be the integral points in V and let $N_V(T)$ be the number of integral solutions x with $||x|| \leq T$. We say V is a "Hardy-Littlewood Variety" if the asymptotic behavior of $N_V(T)$ as $T \to \infty$ is given by a product of local densities which count the number of solutions in V over the real numbers and the p-adic numbers (the exact form of this product of local densities comes from the Hardy-Littlewood method). The known Hardy-Littlewood varieties are ones shown to be so by their method, which requires that n be very large compared to the degrees of the defining equations. However, it appears (from a number of points of view) that to be a Hardy-Littlewood variety is not so restrictive. A junior examined this by looking at a nonsingular cubic form F in six variables, and observed the results predicted by the main arcs.

2.2.4 Prime Spacings

It is a folklore belief that the local spacings between consecutive primes follow the laws of random numbers. A junior investigated both spacings between adjacent primes and spacings between primes in arithmetic progressions for p about 10^{15}, observing excellent agreement with predictions.

2.2.5 Elliptic Curves

An L-function may be associated to an elliptic curve, and one can investigate whether or not the Generalized Riemann Hypothesis holds (namely, all nontrivial zeros should have real part equal to $\frac{1}{2}$). Let m be the order of vanishing of the L-function at the central point $s = \frac{1}{2}$. The Birch and Swinnterton-Dyer conjecture asserts that m equals the order of the Mordell-Weil group. If both conjectures are true, a contour integral of a certain transform of the logarithmic derivative of the L-function (after normalization) should have mean m and error term of a certain size. A junior investigated this for a variety of Elliptic Curves where the rank of the Mordell-Weil group was known. For small rank (rank 0 and 1), good agreement was observed.

2.2.6 $\{n^2\alpha\}$

It is known that for any irrational α, the fractional parts of $n^2\alpha$ for $n \in \mathbf{N}$ are uniformly distributed. However, for most α it is not known whether the distribution of the fractional parts is Poissonian; while it is known that in some special cases the distribution is not Poissonian, it is conjectured that these numbers form a set of measure zero. A junior numerical observed Poissonian distributions for specific α (such as π, e, and $\sqrt{2}$) by examining the consecutive spacing measures, as well as investigating the consecutive spacing measure for an irrational number known not to yield a Poissonian distribution.

3 $2001 - 2002$

3.1 Class Structure

For the first half of the Fall semester, Professor Andrew Wiles and Steven Miller lectured twice a week on background material and basic elliptic curve theory. After all the problems were introduced, the juniors chose their problems. The class continued to meet in small groups, where advanced material for the relevant problems was given in more detail.

Unlike the previous year, all problems shared a common theme, namely an attempt to investigate the observed excess rank phenomenon in families of elliptic curves.

Consider one-parameter families of elliptic curves. By construction, we can force points of infinite order on the curves. Let us assume we have a family where we have forced r points on each curve E_t. By the Birch and Swinnerton-Dyer Conjecture, the geometric rank of the group of rational points should equal the order of vanishing of the associated L-function at the critical point. Assuming the Sign Conjecture, we might expect half the curves to have rank r, and half to have rank $r+1$. (The functional equation being odd / even forces another zero). Previous investigations (see, for example, Fermigier) for many curves with small values for t have shown that this is not the case; ie, that for small t, for many families about 32% have rank r, 48% have rank $r+1$, 18% have rank $r+2$, and 2% have rank $r+3$. Is this excess rank above the forced rank a general phenomenon that persists in the limit, or the result of small numbers?

3.2 Investigations

To numerically investigate the above, the juniors split into several groups. The idea was to study very thoroughly lots of properties of a small number of families, and look for correlations.

Properties studied were the analytic rank (the order of vanishing of the L-function at the critical point), the geometric rank (the order of the Mordell-Weil group), the number of points of low naive and low canonical height, the distribution of the signs of the functional equation, the distribution of $a_t(p)$ (Sato-Tate Conjecture), and the distribution of the first zero above the critical point ($s = \frac{1}{2}$).

For some projects (in particular, the sign investigations) the programs ran very quickly; for others (points of low height, first zero above the critical point) the computation time is significantly longer.

During the Spring semester the juniors and Steve Miller continued to refine the programs. Upon completion, a detailed analysis will be done of the above quantities. The goal is to have a publication for a journal such as Experimental Mathematics, where certain qualitative phenomenon can be reported.

For completeness, a more thorough description of each problem is provided below.

3.2.1 The Sign Conjecture

We can analytically continue the L-function of an elliptic curve, and obtain a functional equation $\Lambda(s, E) = \epsilon_E \Lambda(2-s, E)$, where $\epsilon_E = \pm 1$. The Sign Conjecture for a one-parameter family of elliptic curves asserts that (in the limit) $\epsilon_E = 1$ and $\epsilon_E = -1$ are equally likely. If the family is sufficiently general, we expect the Sign Conjecture to hold.

One student investigated the distribution of signs for 10^7 curves in several such families.

3.2.2 Points of Low Height

For any elliptic curve, we can study the rational solutions $P = (x, y)$. Let $x = \frac{p}{q}$ in lowest terms. The naive height is defined as $h(P) = \max(|p|, |q|)$. Associated to the naive height is the more technical canonical height, which has several useful properties. In particular, the canonical height allows us to determine if rational points are independent.

Juniors calculated all rational solutions with low height for many elliptic curves for two purposes: first, to be used to determine the geometric rank (see below); second, to investigate correlations between the number of points of low height and excess rank. From the Gross-Zagier formula, we know there can be relations between rank and height of points.

3.2.3 Analytic and Geometric Rank

In its simplest form, the Birch and Swinnerton-Dyer Conjecture states the geometric rank of an elliptic curve should equal the analytic rank. Using the canonical height and the lists of observed points of low height, juniors obtained lower bounds for the geometric rank; using standard elliptic curve algorithms (although coding them efficiently was non-trivial), the juniors wrote code to approximate the analytic rank.

All of Fermigier's examples are for one-parameter families where $A_{\mathcal{E}}(p) = \frac{1}{p}\sum_{t(p)} a_t(p)$ is constant. These are natural families to consider, as it is easy to force rational independent points on the curve and create families with rank over $\mathbf{Q}(t)$; however, these may not have the same behavior as a generic one-parameter family. Therefore, more general families will also be investigated.

3.2.4 Sato-Tate

By Hasse, $|a_p| \leq 2\sqrt{p}$; we can therefore write a_p as $2\cos\theta_p$. For curves without complex multiplication, it is conjectured that the angles θ_p are uniformly distributed with respect to the Sato-Tate density, ie, in the limit the percent of curves with angles in $[a, b]$ is $\frac{1}{2\pi}\int_a^b \sin^2 t dt$.

If, however, the elliptic curve has non-zero rank, a lower order bias term should appear in the Sato-Tate conjecture. While this bias will be too small to observe, it is hoped that some of the consequences (such as a predisposition of the $a_t(p)$'s to be negative) will be observable.

3.2.5 Lowest Zero

We expect the zeros of an elliptic curve to follow the GUE distribution. To see this may be beyond the computational ability of the UML (to see the convergence may require more computations than we can do on our machines; many such investigations use large amounts of time on supercomputers). We can, however, look at the first few zeros near the critical point. If we are (again) looking at a family, we can study the distribution of the lowest (or second lowest) zero. We may want to normalize with respect to the logarithm of the conductor.

By looking at functions related to $L(s, E)$, we can construct a real valued function on the line $Re(s) = \frac{1}{2}$ which has the same zeros as $L(s, E)$. By looking for sign changes in this function, we can find the zeros; by standard methods of Complex Analysis, we can do contour integrals to find the number of zeros in a given region. If the number of sign changes equal the contour integral, then we have proved the Generalized Riemann Hypothesis in a region. Note the contour integral is almost certainly not an integer. We will have to approximate it, and then knowing it is an integer will allow us to move from an approximate counting of the number of zeros to an exact counting.

4 Class Resources

Fine 408 is reserved solely for undergraduate use. There are six computers, a blackboard, couch, refrigerator, and work space.

A variety of computational programs and packages are available, including C, Java, Magma, Maple, Mathematica, Matlab, and Pari.

5 Results

Statements of the problems, programs written, and final reports are available online at $http://www.math.princeton.edu/\sim mathlab/index.html.$

6 Future Projects of the UML

In addition to providing undergraduates with research experience in investigating unsolved conjectures, the UML aims to create a database of such research to assist future investigations. Many of last semester's projects naturally lead to future investigations. These future research topics have been noted, and the documentation of the current researcher (programs, papers) will facilitate such future work.

The tentative topic for $2002 - 2003$ is Diophantine Analysis in the Fall (with Professors Ramin Takloo-Bighash and Steven Miller) and Probablity Theory, Random Matrix Theory and Zeros of L-Functions in the Spring (with Professors Yakov Sinai and Steven Miller). Similar to $2001 - 2002$, the idea is to have a unified set of problems. Class size is increasing from 8 and 10 students to 24. We are increasing the staff to two professors, two graduate students, and one undergraduate computer support person. Steven Miller is staying at Princeton for an additional year to help train replacements to continue the program.

7 Contact Information

For questions on the Undergraduate Mathematics Laboratory, please contact either Professor Peter Sarnak (sarnak@math.princeton.edu) or Steven Miller (sjmiller@math.princeton.edu).