# Contents

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CHAPTER 2

# Canonical Models

We saw in Section **??** that the basic objects of the moduli theory of higher dimensional varieties are the canonical models of varieties of general type. The aim of this Chapter is to study these canonical models, and, more generally, log canonical models of pairs.

Section 1 is a summary of the relevant results of the Minimal Model Program or Mori's program. The main result, which we do not prove here, is that for a smoth projective variety of general type $X$, its canonical ring $\sum_{m\geq 0} H^0\bigl(X, \mathcal{O}_X(mK_X)\bigr)$ is finitely generated. Thus its canonical model

$$X^{can} := \operatorname{Proj}_k \sum_{m\geq 0} H^0\bigl(X, \mathcal{O}_X(mK_X)\bigr)$$

is a projective variety.

Examples of the various singularities occurring in the Minimal Model Program are given in Section 2.

Section 3 gives a rather detailed classification of log canonical surface singularities. Strictly speaking, little of it is needed for the general theory, but it is useful and instructive to have a thorough understanding of a class of concrete examples.

Divisorial log terminal singularities are investigated in Section 4. Roughly speaking, these form the largest well-behaved subclass of log canonical singularities. We prove that they are rational and many important sheaves on them are Cohen-Macaulay (120).

Section 5 studies adjunction, a method to relate the properties of a divisor $D \subset X$ to properties of a neighborhood of $D$ in $X$. This is a very important tool that allows induction on the dimension in many cases. The higher codimensional generalizations of these results involve the log canonical centers of a pair $(X, \Delta)$. These are studied in more detail in Section 6.

Cohomological properties of log canonical singularities are investigated in Section 7. The main result (**??**?) implies that if $X$ is proper and log canonical then the natural map

$$H^i(X^{an}, \mathbb{C}) \twoheadrightarrow H^i(X^{an}, \mathcal{O}_{X^{an}}) \quad \text{is surjective with a natural splitting.}$$

This implies that Kodaira vanishing holds for varieties with log canonical singularities and that being CM is deformation invariant for families of stable varieties.

## 1. Canonical singularities and canonical models

### Singularities of the Minimal Model Program.

In this section, our main interest is in varieties over a field of characteristic zero, but everything works over an arbitrary perfect base field. (Some results about surfaces over imperfect fields are in Section 2.)

The definitions and results are all local in the étale or analytic topology, hence they carry over to algebraic and complex analytic spaces.

For most applications, the definition of the canonical class and canonical sheaf given in (**??**) is sufficiently general, but we need the following more general version as well.

Definition 1 (Canonical class and canonical sheaf II.). Let $B$ be a regular base scheme. (In practice, we use only the cases when $B$ is the spectrum of a field or of a DVR.)

Let $X \to B$ be a scheme of finite type over $B$ of pure dimension $n$ that satisfies the following condition:

(1) There is an open subscheme $X^0 \subset X$ and an embedding $\iota : X^0 \hookrightarrow \mathbb{P}^N_B$ such that
   (a) $Z := X \setminus X^0$ has codimension $\geq 2$ in $X$, and
   (b) $\iota(X^0)$ is a local complete intersection in $\mathbb{P}^N_B$.

(In this book we use only three special cases of this. First, if $X$ is normal and quasi-projective then $Z = \operatorname{Sing} X$ works. Second, in dealing with stable varieties, we consider schemes $X$ that have ordinary nodes at some codimension 1 points. Finally. we occasionally use the dualizing sheaf for nonreduced divisors in a nonsingular scheme.)

Let $I$ denote the ideal sheaf of the closure of $\iota(X^0)$. Then $I/I^2$ is a locally free sheaf on $\iota(X^0)$ and, as in [**Har77**, 8.20], we set

$$\omega_{X^0} := \iota^*\big(\omega_{\mathbb{P}^N_B} \otimes \det^{-1}\big(I/I^2\big)\big). \tag{1.2}$$

Finally define the *canonical sheaf* of $X$ as

$$\omega_X := j_*\omega_{X^0}, \tag{1.3}$$

where $j : X^0 \hookrightarrow X$ denotes the open embedding. (Strictly speaking, one should indicate $B$ and denote it by $\omega_{X/B}$ instead. However, most of the time we can only handle the case when $B$ is the spectrum of a field of charactersitic 0, thus there is little to gain by adding $B$ to the notation.) If $X$ is reduced, the corresponding linear equivalence class of Weil divisors is denoted by $K_X$. (If $X^0$ is smooth over $B$, then one can define $\omega_X$ using differentials as in (**??**). In general, differential forms give a different sheaf.)

Note that while [**Har77**, 8.20] is a theorem, for us (1.2–3) are definitions. Therefore we need to establish that $\omega_X$ does not depend on the projective embedding chosen. This is easy to do by comparing two different embeddings $\iota_1, \iota_2$ with the diagonal embedding

$$(\iota_1, \iota_2) : X^0 \hookrightarrow \mathbb{P}^{N_1}_B \times_B \mathbb{P}^{N_2}_B.$$

We also need that $\omega_X$ is the relative *dualizing sheaf* $\omega_{X/B}$. If $X$ itself is projective, a relatively short discussion is in [**KM98**, Sec.5.5]. For the general case see [**Har66, Con00**].)

Definition 2 (Discrepancy I.). Let $X$ be a normal variety over a field $k$ such that $mK_X$ is Cartier for some $m > 0$. Suppose $f : Y \to X$ is a birational morphism from a smooth variety $Y$. (We do not assume that $f$ is proper. Thus we usually start with a proper birational morphism $f^* : Y^* \to X$ and let $Y \subset Y^*$ denote the open subscheme of smooth points.) Let $E \subset Y$ be an irreducible divisor and $e \in E$

a general point of $E$. Let $\{y_i\}$ be a local coordinate system at $e \in Y$ such that $E = (y_1 = 0)$. Then, locally near $e$,

$$f^*\big(\text{local generator of } \mathcal{O}_X(mK_X) \text{ at } f(e)\big) = y_1^{c(E,X)} \cdot (\text{unit}) \cdot (dy_1 \wedge \cdots \wedge dy_n)^{\otimes m}$$

for some integer $c(E,X)$. The rational number $a(E,X) := \frac{1}{m}c(E,X)$ is called the *discrepancy* of $E$ with respect to $X$. It is independent of the choice of $m$.

We refer to any such $E$ as a *divisor over* $X$. The closure of $f(E) \subset Y$ is called the *center* of $E$ on $X$. It is denoted by $\operatorname{center}_X E$.

Assume that $f' : Y' \to X$ is another birational morphism and $E' \subset Y'$ an irreducible divisor such that the rational map $f^{-1} \circ f' : Y' \to X \dashrightarrow Y$ is an isomorphism at the general points $e \in E$ and $e' \in E'$. Then we see from the definition that $a(E,X) = a(E',X)$ and $\operatorname{center}_X E = \operatorname{center}_X E'$. Because of this, in discrepancy considerations, we frequently do not distinguish $E$ from $E'$.

If $f : Y \to X$ is a birational morphism then $mK_Y$ is linearly equivalent to

$$f^*(mK_X) + \sum_i (m \cdot a(E_i,X))E_i,$$

where the $E_i$ are the $f$-exceptional divisors. We can formally divide by $m$ and write

$$K_Y \sim_{\mathbb{Q}} f^*K_X + \sum a(E_i,X)E_i.$$

A basic property of discrepancy is that it is positive when $X$ is smooth.

Proposition 3. *Let $X$ be a smooth variety over a field $k$. Then $a(E,X) \geq 1$ for every exceptional divisor $E$ over $X$.*

Proof. Let $f : Y \to X$ be a birational morphism, $Y$ normal, $E \subset Y$ an exceptional divisor and $e \in E$ a general point. Choose local coordinates $(y_1, \ldots, y_n)$ near $e \in Y$ and $(x_1, \ldots, x_n)$ near $f(e) \in X$. Then

$$f^*\big(dx_1 \wedge \cdots \wedge dx_n\big) = \operatorname{Jac}\Big(\frac{x_1, \ldots, x_n}{y_1, \ldots, y_n}\Big) dy_1 \wedge \cdots \wedge dy_n,$$

thus $a(E,X)$ is the order of vanishing of the Jacobian along $E$. Hence $a(E,X) \geq 0$ and, by the inverse function theorem, $a(E,X) = 0$ iff $f$ is locally invertible at $f(e)$. Thus $a(E,X) \geq 1$ for every exceptional divisor $E$. $\square$

Notation 4. Let $X$ be a normal scheme. For a Weil divisor $D$, $\mathcal{O}_X(D)$ is a rank 1 reflexive sheaf. The correspondance $D \mapsto \mathcal{O}_X(D)$ is an isomorphism from the group $\operatorname{Cl}(X)$ of Weil divisors modulo linear equivalence to the group of rank 1 reflexive sheaves.

The product of two reflexive sheaves $L_1, L_2$ is given by $\big(L_1 \otimes L_2\big)^{**}$, the double dual or reflexive hull of the usual tensor product. For powers we use the notation

$$L^{[m]} := \big(L^{\otimes m}\big)^{**}.$$

Note further that if $L$ is a reflexive sheaf and $D = \sum a_i D_i$ a Weil divisor then $L(D)$ denotes the sheaf of rational sections of $L$ with poles of multiplicity at most $a_i D_i$. It is thus the double dual of $L \otimes \mathcal{O}_X(D)$.

More generally, let $X$ be a reduced, pure dimensional scheme. Let $\operatorname{Cl}^*(X)$ denote the group of Weil divisors, none of whose irreducible components are contained in $\operatorname{Sing} X$, modulo linear equivalence. (Thus, if $X$ is normal, then $\operatorname{Cl}^*(X) = \operatorname{Cl}(X)$.) As before, $D \mapsto \mathcal{O}_X(D)$ is an isomorphism from $\operatorname{Cl}^*(X)$ to the group of rank 1 reflexive sheaves that are locally free at all codimension 1 points of $X$.

The definition (2) can be generalized to pairs $(X, \Delta)$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier:

5 (Pairs). Mori's program was originally conceived to deal with smooth projective varieties. Later it became clear that one needs to handle certain singular varieties and also add a divisor to the basic object. Next we define a rather general set-up where the basic definitions make sense and fundamental results hold.

*Assumption* 5.1. Our main interest is in pairs $(X, \Delta)$ where $X$ is a normal variety over a field $k$ and $\Delta = \sum a_i D_i$ is a linear combination of distinct prime divisors. We allow the $a_i$ to be arbitrary rational numbers.

*Assumption* 5.2. More generally, we also consider pairs $(X, \Delta)$ where $X$ is a reduced, pure dimensional scheme of finite type over a regular base scheme $B$ satsifying (1.1) and $\Delta = \sum a_i D_i$ is a linear combination of distinct prime divisors none of which is contained in $\operatorname{Sing} X$. As in (1), we write $\omega_X$ and $K_X$ instead of $\omega_{X/B}$ and $K_{X/B}$.

Note that if $ma_i$ is an integer for every $i$ then $\omega_X^{[m]}(m\Delta)$ is locally free outside a codimension 2 subset $Z \subset X$.

Definition 6 (Discrepancy II.). Let $(X, \Delta)$ be a pair as in (5). Assume that $m(K_X + \Delta)$ is Cartier for some $m > 0$. Equivalently, $m\Delta$ has integral coefficients and $\omega_X^{[m]}(m\Delta)$ is locally free.

Suppose $f : Y \to X$ is a birational morphism from a nonsingular scheme $Y$. Let $E \subset Y$ denote the exceptional locus of $f$ and $E_i \subset E$ the irreducible exceptional divisors. Let
$$f_*^{-1}\Delta := \sum a_i f_*^{-1} D_i$$
denote the birational transform of $\Delta$. There is a natural isomorphism of invertible sheaves
$$\iota_{Y \setminus E} : \omega_Y^{[m]}\big(m f_*^{-1}\Delta\big)|_{Y\setminus E} \cong f^*\big(\omega_X^{[m]}(m\Delta)\big)|_{Y \setminus E}. \tag{6.1}$$
Thus there are rational numbers $a(E_i, X, \Delta)$ such that $m \cdot a(E_i, X, \Delta)$ are integers, and $\iota_{Y\setminus E}$ extends to an isomorphism
$$\iota_Y : \omega_Y^{[m]}\big(m f_*^{-1}\Delta\big) \cong f^*\big(\omega_X^{[m]}(m\Delta)\big)\big(\textstyle\sum_i m \cdot a(E_i, X, \Delta) E_i\big). \tag{6.2}$$
By definition $a(D_i, X, \Delta) = -a_i$ and $a(D, X, \Delta) = 0$ for any divisor $D \subset X$ which is different from the $D_i$. The rational number $a(E, X, \Delta)$ is called the *discrepancy* of $E$ with respect to $(X, \Delta)$. As in the $\Delta = 0$ case, $a(E_i, X, \Delta)$ depends only on $E_i$ but not on $f$.

Notation 7. The pull-back and the discrepancies can be conveniently packaged into any of the 3 equivalent forms:
$$\begin{array}{lll} K_Y + f_*^{-1}\Delta & \sim_{\mathbb{Q}} & f^*(K_X + \Delta) + \sum_{E_i:\text{exceptional}} a(E_i, X, \Delta) E_i, \quad \text{or} \\ K_Y & \sim_{\mathbb{Q}} & f^*(K_X + \Delta) + \sum_{E_i:\text{arbitrary}} a(E_i, X, \Delta) E_i, \quad \text{or} \\ K_Y + \Delta_Y & \sim_{\mathbb{Q}} & f^*(K_X + \Delta) \quad \text{where } f_*\Delta_Y = \Delta. \end{array}$$
Note, however, that these formulas do not show that the isomorphisms in (6.1–2) are canonical.

We frequently refer to these formulas by saying, for instance: "write $K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + A$." In this case it is understood that $A$ is chosen as above. That is, we have to make sure that $A = \sum_i a(E_i, X, \Delta) E_i$. It is very useful to know that we need to check this only for non-exceptional divisors since, essentially by the Hodge index theorem (31), a numerically trivial exceptional divisor is in fact trivial.

8 (Real coefficients). One can also define discrepancies for pairs $(X, \Delta)$ where $X$ is a normal variety and $\Delta = \sum a_i D_i$ is a linear combination of distinct prime divisors with real coefficients, as long as the pull-back $f^*(K_X + \Delta)$ can be defined. The latter holds of $K_X + \Delta$ is a linear combination of Cartier divisors with real coefficients. (Unlike in the rational case, this is weaker than assuming that a real multiple of $K_X + \Delta$ be Cartier.) If $(X, \Delta)$ is lc (or klt, ...) then there are arbitrarily small rational perturbations $\Delta'$ of $\Delta$ such that $K_X + \Delta'$ is $\mathbb{Q}$-Cartier and $(X, \Delta')$ is also lc (or klt, ...). All the results of this section work for real divisors. See [**BCHM06**] or [**Kol08**, Sec.4] for some foundational issues.

The basic example is the following "log smooth" version of (3).

PROPOSITION 9. *Let $X$ be a smooth variety over a field $k$ and $\Delta = \sum D_i$ a nc divisor. Then $a(E, X, \Delta) \geq -1$ for every divisor $E$ over $X$.*

Proof. Let $f : Y \to X$ be any birational morphism, $Y$ smooth. Let $E \subset Y$ be an exceptional divisor, $e \in E$ a general point and choose local coordinates $(y_1, \dots, y_n)$ such that $E = (y_1 = 0)$.

Assume first that $(X, \Delta)$ is snc near $f(e)$. Let $(x_1, \dots, x_n)$ be local coordinates on a neighborhood $U \ni f(e)$ such that $\Delta \cap U = (x_1 \cdots x_r = 0)$. A local generator of $\mathcal{O}_U(K_U + \Delta)$ is given by

$$\sigma_U := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{x_r} \wedge dx_{r+1} \wedge \cdots \wedge dx_n.$$

Near $e$, we can write $f^* x_i = y_1^{a_i} \cdot u_i$ where $u_i$ is a unit. Thus

$$f^* \frac{dx_i}{x_i} = \frac{d(y_1^{a_i} \cdot u_i)}{y_1^{a_i} \cdot u_i} = a_i \frac{dy_1}{y_1} + \frac{du_i}{u_i}.$$

Since $dy_1 \wedge dy_1 = 0$, we obtain that

$$f^* \sigma_U = \frac{dy_1}{y_1} \wedge \omega_{n-1} + \omega_n,$$

where $\omega_{n-1}$ is a regular $(n-1)$-form and $\omega_n$ is a regular $n$-form. Thus $f^* \sigma_U$ is a section of $\mathcal{O}_Y(K_Y + E)$ near $E$ and so $a(E, X, \Delta) \geq -1$.

The general nc case is obtained by using a suitable étale neighborhood of $f(e)$ instead of a Zariski neighborhood. □

The discrepancies measure the singularities of a pair $(X, \Delta)$ together. Large discrepancies indicate that $(X, \Delta)$ is mildly singular and negative values indicate that $(X, \Delta)$ is more singular.

When $\Delta = 0$, we measure the singularities of $X$ and when $X$ is smooth, we measure the singularities of $\Delta$, but the interplay between $X$ an $\Delta$ is quite subtle. In general, the discrepancies have the following obvious monotonicity property:

LEMMA 10. *Notation as in (6). Assume that $\Delta'$ is effective and $\mathbb{Q}$-Cartier. Then $a(E, X, \Delta) \geq a(E, X, \Delta + \Delta')$ for every divisor $E$ over $X$, and strict inequality holds iff* $\operatorname{center}_X E \subset \operatorname{Supp} \Delta'$. □

11 (Regular schemes). The basic lemmas (3) and (9) both hold for regular schemes $X$ as in (1), but they need a different proof since differential forms do not give sections of $\omega_X$.

The easiest is to compute first what happens if we blow up a regular subvariety $Z \subset X$. The formula given in [**Har77**, Exrc.II.8.5] is correct in general and the proof suggested there also works. (The set up in [**Har77**, Exrc.II.8.5] tacitly assumes that $X$ is over a perfect field.) As in [**KM98**, 2.31], by induction this gives the result for all exceptional divisors using [**KM98**, 2.45].

The following lemma reduces, in principle, any discrepancy computation to the case when $X$ is smooth and $\operatorname{Supp}\Delta$ is a snc divisor (assuming that resolution of singularities holds).

LEMMA 12. [**KM98**, 2.30] *Let $f : Y \to X$ be a proper birational morphism. Let $\Delta_Y$ (resp. $\Delta_X$) be $\mathbb{Q}$-divisors on $Y$ (resp. $X$) such that*

$$K_Y + \Delta_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta_X) \quad \textit{and} \quad f_*\Delta_Y = \Delta_X.$$

*Then, for any divisor $F$ over $X$,*

$$a(F, Y, \Delta_Y) = a(F, X, \Delta_X). \quad \square$$

Note that even if we are interested in the $\Delta_X = 0$ case, $\Delta_Y$ is almost always nonzero and contains divisors both with positive and negative coefficients.

Next we define the 6 classes of singularities that are most important for the minimal model program.

DEFINITION 13. Let $(X, \Delta)$ be a pair where $X$ is a normal scheme of dimension $\geq 2$ and $\Delta = \sum a_i D_i$ is a formal sum of distinct prime divisors. (We allow the $a_i$ to be arbitrary rational numbers.) Assume that $m(K_X + \Delta)$ is Cartier for some $m > 0$. We say that $(X, \Delta)$ is

$$\left.\begin{array}{c} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{dlt} \\ \textit{lc} \end{array}\right\} \quad \text{if } a(E, X, \Delta) \text{ is} \quad \left\{\begin{array}{ll} > 0 & \text{for every exceptional } E, \\ \geq 0 & \text{for every exceptional } E, \\ > -1 & \text{for every } E, \\ > -1 & \text{for every exceptional } E, \\ > -1 & \text{if } \operatorname{center}_X E \subset \text{non-snc}(X, \Delta), \\ \geq -1 & \text{for every } E. \end{array}\right.$$

Here klt is short for *"Kawamata log terminal"*, plt for *"purely log terminal"*, dlt for *"divisorial log terminal"* and lc for *"log canonical"*. The set of points where $(X, \Delta)$ is not snc is denoted by non-snc$(X, \Delta)$.

(The frequently used phrase "$(X, \Delta)$ has terminal etc. singularities" may be confusing since it could refer to the singularities of $(X, 0)$ instead.)

Each class contains the previous one, except canonical does not imply klt if $\Delta$ contains a divisor with coefficent 1. The key point is to show that if $a(E, X, \Delta) \geq -1$ for every exceptional $E$ then $a(E, X, \Delta) \geq -1$ for every $E$ [**KM98**, 2.31]. This last claim fails if $\dim X = 1$ since there are no exceptional divisors at all. If $\dim X = 1$ then $(X, \sum a_i D_i)$ is terminal/klt iff $a_i < 1$ for every $i$. The other 4 concepts all coincide and they hold iff $a_i \leq 1$ for every $i$.

*Warning on effectivity.* In final applications, the above concepts are useful only if $\Delta$ is an effective divisor, and in the literature, frequently the definitions assume that $\Delta \geq 0$. However, in some inductive proofs, it is very convenient to allow $\Delta$ to contain some divisors with negative coefficients. (See (12) for a typical example.) The usage is inconsistent in the literature, probably even in this book.

Each of these 6 notions has an important place in the theory of minimal models:

(1) Terminal: Assuming $\Delta = 0$, this is the smallest class that is necessary to run the minimal model program for smooth varieties. The $\Delta \neq 0$ case appears only infrequently.
(2) Canonical: Assuming $\Delta = 0$, these are precisely the singularities that appear on the canonical models of varieties of general type. This class is especially important for moduli problems.
(3) Kawamata log terminal: The proofs of the vanishing theorems seem to run naturally in this class but it is not suitable for inductive proofs.
    If $\Delta = 0$ then the notions klt, plt and dlt coincide and in this case we say that $X$ has *log terminal* (abbreviated to *lt*) singularities.
    This class is also easy to connect to analysis. If $M$ is a smooth complex manifold and $f$ is a meromorphic function and $\Delta := (f = 0) - (f = \infty)$, then $(M, c\Delta)$ is klt iff $|f|^{-c}$ is locally $L^2$, see [**Kol97**, 3.2].
(4) Purely log terminal: This class was invented for inductive purposes. We do not use it much.
(5) Divisorial log terminal: These are the singularities we obtain if we start with an snc pair $(X, \Delta)$ and run the MMP. These are much better behaved than log canonical pairs.
(6) Log canonical: This is the largest class where discrepancy still makes sense. It contains many cases that are rather complicated from the cohomological point of view. Therefore it is quite hard to work with. However, these singularities appear naturally on the stable varieties at the boundary of our moduli spaces, hence they can not be ignored.

For basic examples illustrating the nature of these singularities see Section 2.

Given $(X, \Delta)$, the most important value for us is the minimum of $a(E, X, \Delta)$ as $E$ runs through various sets of divisors. We use several versions:

DEFINITION 14. The *discrepancy* of $(X, \Delta)$ is given by

$$\operatorname{discrep}(X, \Delta) := \inf_E \{a(E, X, \Delta) : E \text{ is an exceptional divisor over } X\ \}.$$

(That is, $E$ runs through all the irreducible exceptional divisors of all birational morphisms $f : Y \to X$.)

The *total discrepancy* of $(X, \Delta)$ is defined as

$$\operatorname{totaldiscrep}(X, \Delta) := \inf_E \{a(E, X, \Delta) : E \text{ is a divisor over } X\ \}.$$

(That is, $E \subset Y$ runs through all the irreducible exceptional divisors for all birational morphisms $f : Y \to X$ and through all the irreducible divisors of $X$.)

One problem with the above definitions is that one needs to check discrepancies on all possible birational maps to $X$. By (12), the computation of discrepancies can be reduced to smooth varieties with snc divisors. In the latter case, there is an explicit formula:

LEMMA 15. [**KM98**, 2.31] *Let $X$ be a smooth variety and $\Delta = \sum a_i D_i$ a nc divisor. Assume that $a_i \leq 1$ for every $i$. Then*

$$\operatorname{discrep}(X, \Delta) = \min\Big\{1, \min_i\{1 - a_i\}, \min_{D_i \cap D_j \neq \emptyset}\{1 - a_i - a_j\}\Big\}.$$

*In particular,* $-1 \leq \operatorname{discrep}(X, \Delta) \leq 1$ *and*

$$\operatorname{discrep}\big(X, \textstyle\sum_i a_i D_i\big) = \operatorname{discrep}\big(X, \textstyle\sum_{i: a_i > 0} a_i D_i\big). \quad \square$$

Corollary 16. *Let $X$ be a normal variety over a perfect field $k$ such that $K_X$ is $\mathbb{Q}$-Cartier and $f : Y \to X$ a resolution of singularities. Write $K_Y \sim_{\mathbb{Q}} f^*K_X + \sum a_i E_i$ where the sum runs over all $f$-exceptional divisors $E_i$. Then*

(1) *$X$ is canonical iff $a_i \geq 0$ for every $i$.*
(2) *$X$ is terminal iff $a_i > 0$ for every $i$.*
(3) *If $X$ is canonical then* $\operatorname{discrep}(X) = \min\{1, \min_i\{a_i\}\}$.

Proof. Since $a(E_i, X) = a_i$, the conditions are necessary. Conversely, assume that $a_i \geq 0$. If $F$ is an exceptional divisor of $f$ then $a(F, X)$ equals the coefficient of $F$ in $\sum a_i E_i$, hence positive by assumption. If $F$ is any divisor that is exceptional over $Y$, then, by (12), $a(F, X) = a(F, Y, -\sum a_i E_i)$ and, using (10) and (3) we obtain that $a(F, Y, -\sum a_i E_i) \geq a(F, Y) \geq 1$. □

The proof of the version with boundary is the same:

Corollary 17. *Let $X$ be a normal variety and $\Delta = \sum d_j D_j$ a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log-resolution of singularities. Write $K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum a_i E_i$. Then*

(1) *$(X, \Delta)$ is log canonical iff $a_i \geq -1$ for every $i$.*
(2) *$(X, \Delta)$ is klt iff $a_i > -1$ for every $i$.* □

(Note that by our conventions the birational transforms $f_*^{-1}(D_j)$ are among the $E_i$ with coefficient $a_i = -d_j$. Thus the restrictions on the $a_i$ imply that $d_j \leq 1$ (resp. $d_j < 1$) for every $j$. A formula for $\operatorname{discrep}(X, \Delta)$ is in [**KM98**, 2.32])

In (**??**) we saw examples of singular rational surfaces whose canonical class is ample. Thus, for singular varieties, the plurigenera are not birational invariants. The following result shows that canonical singularities form the largest class where the plurigenera are birational invariants.

Proposition 18. *Let $X$ be a normal projective variety such that $K_X$ is $\mathbb{Q}$-Cartier and ample. Let $f : Y \dashrightarrow X$ be a birational map from a nonsingular proper variety to $X$. Then $X$ has canonical singularities iff*

$$H^0(X, \mathcal{O}_X(mK_X)) = H^0(Y, \mathcal{O}_Y(mK_Y)) \quad \textit{for every } m \geq 0.$$

Proof. Let $Y'$ be a normal, proper variety and $Y' \to Y$ a birational morphism such that the composite $g : Y' \to Z$ is a morphism.

Pick $m$ such that $mK_X$ is Cartier and write $mK_{Y'} \sim g^*(mK_X) + A$ where $A$ is $g$-exceptional. Then

$$\begin{aligned} H^0\big(Y, \mathcal{O}_Y(mK_Y)\big) &= H^0\big(Y', \mathcal{O}_{Y'}(mK_{Y'})\big) \\ &= H^0\big(X, g_*\big(\mathcal{O}_{Y'}(mK_{Y'})\big)\big) \\ &= H^0\big(X, \mathcal{O}_X(mK_X) \otimes g_*\mathcal{O}_{Y'}(A)\big). \end{aligned}$$

If $A \geq 0$ then $g_*\mathcal{O}_{Y'}(A) = \mathcal{O}_X$ hence the last term equals $H^0(X, \mathcal{O}_X(mK_X))$. If $A$ is not effective then $g_*\mathcal{O}_{Y'}(A) \subsetneq \mathcal{O}_X$, thus, for $m \gg 1$,

$$H^0\big(Y, \mathcal{O}_Y(mK_Y)\big) = H^0\big(Y', \mathcal{O}_{Y'}(mK_{Y'})\big) \subsetneq H^0\big(X, \mathcal{O}_X(mK_X)\big).$$

This takes care of all sufficiently divisible values of $m$. The rest follows from (20). □

An essentially identical proof, using $n$-forms with poles along divisors shows the following:

PROPOSITION 19. *Let $X$ be a normal projective variety and $D$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier and ample. Let $f : Y \dashrightarrow X$ be a birational map from a nonsingular proper variety to $X$ and $D_Y$ a nc divisor on $Y$ such that $f_*(D_Y) = D$ and every irreducible component of* $\mathrm{Ex}(f)$ *appears in $D_Y$ with coefficient 1. Then $(X, D)$ is log-canonical iff*

$$H^0\bigl(X, \mathcal{O}_X(mK_X + \lfloor mD \rfloor)\bigr) = H^0\bigl(Y, \mathcal{O}_Y(mK_Y + \lfloor mD_Y \rfloor)\bigr) \quad \textit{for every } m \geq 0. \qquad \square$$

LEMMA 20. *Let $X$ be a normal variety, $D$ a $\mathbb{Q}$-divisor and $m$ a positive integer such that $mD$ is an integral divisor. Let $s$ be a rational section of $\mathcal{O}_X(\lfloor D \rfloor)$. Then $s$ is a regular section of $\mathcal{O}_X(\lfloor D \rfloor)$ iff $s^m$ is a regular section of $\mathcal{O}_X(mD)$.*

Proof. Since $X$ is normal, it is enough to check this at the generic point of every divisor. So pick a prime divisor $E \subset X$ and assume that $E$ has coefficient $r$ in $D$. Let $s$ have a pole of order $n$ along $E$. Then $s$ is a regular section of $\mathcal{O}_X(\lfloor D \rfloor)$ along $E$ iff $n \leq \lfloor r \rfloor$ and $s^m$ is a regular section of $\mathcal{O}_X(mD)$ along $E$ iff $mn \leq mr$. Since $n$ is an integer, these are equivalent. $\square$

**Minimal and Canonical Models.**

21 (Canonical models). Let $X$ be a smooth projective variety over a field $k$. Its *canonical ring* is the graded ring

$$R(X) = R(X, K_X) := \sum_{m \geq 0} H^0\bigl(X, \mathcal{O}_X(mK_X)\bigr).$$

The canonical ring depends only on the birational equivalence class of $X$, and conversely, for varieties of general type, the canonical ring determines the birational equivalence class of $X$.

If the canonical ring is finitely generated, then

$$X^{\mathrm{can}} := \mathrm{Proj}_k\, R(X, K_X)$$

is called the *canonical model* of $X$ (or of its birational equivalence class).

For various reasons, we are also interested in *minimal models* of $X$. These are varieties $X^{\min}$ that are birational to $X$, have terminal singularities and nef canonical class. More generally, $X^w$ is a *weak canonical model* of $X$ if $X^w$ has canonical singularities, nef canonical class and is birational to $X$. Thus canonical and minimal models are also weak canonical models.

THEOREM 22. [**Rei80**] *Let $X$ be a smooth, proper variety of general type such that its canonical ring $R(X, K_X)$ is finitely generated. Then*

(1) *its canonical model $X^{\mathrm{can}}$ is normal, projective, birational to $X$,*
(2) *the canonical class $K_{X^{\mathrm{can}}}$ is $\mathbb{Q}$-Cartier and ample,*
(3) *$X^{\mathrm{can}}$ has canonical singularities, and*
(4) *$H^0\bigl(X, \mathcal{O}_X(mK_X)\bigr) \cong H^0\bigl(X^{\mathrm{can}}, \mathcal{O}_{X^{\mathrm{can}}}(mK_{X^{\mathrm{can}}})\bigr)$ for every $m \geq 0$.*

Proof. Since $\sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$ is finitely generated, there is an $r > 0$ such that $\sum_{m \geq 0} H^0(X, \mathcal{O}_X(mrK_X))$ is generated by $H^0(X, \mathcal{O}_X(rK_X))$. Thus $D := rK_X$ satisfies the assumptions of (23). Therefore, there is a birational map $\phi : X \dashrightarrow Z$ to a normal variety and a very ample Cartier divisor $H$ on $Z$ such that $H \sim \phi_*(D)$ and $\phi^* : H^0(Z, \mathcal{O}_Z(mH)) \to H^0(X, \mathcal{O}_X(mD))$ is an isomorphism for every $m > 0$. Thus $Z = X^{\mathrm{can}}$.

Since the push forward of the canonical class by a birational map is the canonical class, we see that

$$rK_Z \sim \phi_*(rK_X) \sim \phi_*(D) \sim H.$$

Thus $K_Z$ is $\mathbb{Q}$-Cartier and ample. $Z$ has canonical singularities by (18). □

PROPOSITION 23. *Let $X$ be a proper, normal variety and $D$ a Cartier divisor on $X$. Assume that $\sum_m H^0\big(X, \mathcal{O}_X(mD)\big)$ is generated by $H^0\big(X, \mathcal{O}_X(D)\big)$ and that $h^0\big(X, \mathcal{O}_X(mD)\big) > cm^{\dim X}$ for some $c > 0$. Let $\phi : X \dashrightarrow \mathbb{P}^N$ denote the map given by $|D|$. Let $Z$ be the closure of $\phi(X)$ and $|H|$ the hyperplane class on $Z$. Then*

(1) *$\phi$ is birational,*
(2) *$Z$ is a normal,*
(3) *$Z \setminus \phi\big(X \setminus \operatorname{Bs}|D|\big)$ has codimension $\geq 2$ in $Z$,*
(4) *every divisor in $\operatorname{Bs}|D|$ is contracted by $\phi$ and*
(5) *$\phi_*|D| = |H|$.*

Proof. Let $X \leftarrow X' \to Z$ be the normalization of the closure of the graph of $\phi$ with projections $\pi$ and $\phi'$. Set $D' := \pi^* D$. Then $X'$ and $D'$ satisfy our assumptions. Moreover, $\operatorname{Bs}|D'| = \pi^{-1}\operatorname{Bs}|D|$ implies that it is enough to show the conclusions for $X'$ and $D'$. To simplify notation, assume from now on that $\phi$ is a morphism, $|D| = \phi^*|H| + F$ and $F$ is the base locus of $|D|$. Since $\sum_m H^0\big(X, \mathcal{O}_X(mD)\big)$ is generated by $H^0\big(X, \mathcal{O}_X(D)\big)$, we conclude that

$$H^0\big(Z, \mathcal{O}_Z(mH)\big) = H^0\big(X, \mathcal{O}_X(mD)\big) > cm^{\dim X}.$$

In particular, $\dim Z = \dim X$ and $\phi$ is generically finite.

Let $p : Z' \to Z$ be the normalization of $Z$ in $X$. For large $m$, $m(p^*H)$ is very ample on $Z'$, but

$$H^0\big(Z, \mathcal{O}_Z(mH)\big) \subset H^0\big(Z', \mathcal{O}_{Z'}(m(p^*H))\big) \subset H^0\big(X, \mathcal{O}_X(mD)\big) = H^0\big(Z, \mathcal{O}_Z(mH)\big).$$

Thus $Z = Z'$ is normal, proving (1) and (2).

Let $B$ be any irreducible divisor on $X$ such that $\phi(B)$ is a divisor. We can view $\mathcal{O}_Z\big(\phi(B)\big)$ as a subsheaf of the sheaf of rational functions. It has thus an inverse image $\mathcal{O}_Z\big(\phi(B)\big)\cdot\mathcal{O}_X$ which is a coherent subsheaf of the sheaf of rational functions on $X$. Thus there is a $\phi$-exceptional divisor $E$ such that

$$\mathcal{O}_Z\big(\phi(B)\big)\cdot\mathcal{O}_X \subset \mathcal{O}_X(B+E).$$

Let $H_0$ be a Cartier divisor on $Z$ that vanishes along $\phi(E)$ but nor along $\phi(B)$. Then, for some $m_0 > 0$,

$$\mathcal{O}_Z\big(\phi(B) - m_0H_0\big)\cdot\mathcal{O}_X \subset \mathcal{O}_X(B+E-m_0\phi^*H_0) \subset \mathcal{O}_X(B)$$

and these 3 sheaves agree generically along $B$.

Take now $m \gg 1$ such that $\mathcal{O}_Z\big(mH + \phi(B) - m_0H_0\big)$ is generated by global sections. Then $\mathcal{O}_X(m\phi^*H + B)$ has a global section that does not vanish along $B$. In particular, such a $B$ is not in the base locus of $|m\phi^*H + B|$. Since $|mD| = |m\phi^*H| + mF$, we see that $B$ is not in $F = \operatorname{Bs}|D|$.

Thus $\phi\big(\operatorname{Bs}|D|\big)$ has codimension $\geq 2$ in $Z$, which implies (3) and (4). Finally $\phi_*|D| = \phi_*\phi^*|H| + \phi_*F = |H|$, proving (5). □

**Canonical Models of Pairs.**

Next we generalize these notions to pairs $(X, \Delta)$ and to the relative setting. The guiding principle is that the *(log) canonical ring*

$$R(X, K_X + \Delta) := \sum_{m \geq 0} H^0(X, mK_X + \lfloor m\Delta \rfloor)$$

should play the role of the canonical ring.

*Aside.* Note that $\lfloor A + B \rfloor \geq \lfloor A \rfloor + \lfloor B \rfloor$ for any divisors $A, B$, thus, for any divisor $D$, $R(X, D) := \sum_{m \geq 0} H^0(X, \lfloor mD \rfloor)$ is a ring. In particular, the canonical ring is indeed a ring. On the other hand, $R^u(X, D) := \sum_{m \geq 0} H^0(X, \lceil mD \rceil)$ is, in general, not a ring. However, $\lceil A + B \rceil \geq \lfloor A \rfloor + \lceil B \rceil$, thus $R^u(X, D)$ is an $R(X, D)$-module.

A new twist is that while it is straightforward to define when $(X^c, \Delta^c)$ is a canonical model, it is harder to pin down when is $(X^c, \Delta^c)$ a canonical model of another pair $(X, \Delta)$. The main reason is that $(X^c, \Delta^c)$ carries no information on those irreducible components of $\Delta$ which are exceptional for $X \dashrightarrow X^c$.

Given a pair $(X, \Delta)$, what should its canonical model be?

First of all, it is a pair $(X^c, \Delta^c)$ which is canonical (24). Second, $X^c$ should be birational to $X$. More precisely, there should be a birational map $\phi : X \dashrightarrow X^c$ which is a contraction. (That is, $\phi^{-1}$ does not contract any divisors.) Then the only sensible choice is to set $\Delta^c := \phi_* \Delta$.

However, these conditions are not yet sufficient, as shown by the next example.

Let $(Y, 0)$ be log canonical with ample $K_Y$ and $f : X \to Y$ a resolution with exceptional divisors $E_i$. Write

$$f^* K_Y \sim_{\mathbb{Q}} K_X + \sum_i b_i E_i \quad \text{where } b_i = -a(E_i, X, 0).$$

For some $0 \leq c_i \leq 1$, set $\Delta_X := \sum c_i E_i$. Then $f_* \Delta_X = 0$. As in (18) and (19), we see that the two rings

$$\textstyle\sum_{m \geq 0} H^0(Y, mK_Y) \quad \text{and} \quad \sum_{m \geq 0} H^0\big(X, mK_X + \lfloor m\Delta_X \rfloor\big)$$

agree only if $\Delta_X \geq \sum_i b_i E_i$, that is, if $c_i \geq b_i$ for every $i$.

Not that this problem did not occur in the canonical case. Indeed, if $(X, 0)$ is canonical then $b_i = -a(E_i, X, 0) \leq 0$ and on $X$ we take $\Delta_X = 0$. Thus $\Delta_X \geq \sum_i b_i E_i$ is automatic.

By contrast, in the log canonical case, we only assume that $b_i = -a(E_i, X, 0) \leq 1$, hence the condition $c_i \geq b_i$ is nontrivial.

Keeping this example in mind, we see that we have to compare the discrepancies of divisors with respect to the canonical model and the discrepancies with respect to the original pair.

The discrepancy inequality should hold for all divisors, but it turns out (see (30)) that it is enough to require it for $\phi$-exceptional divisors (26.5).

Eventually we also need these concepts in the relative setting, that is, over rather general base schemes.

DEFINITION 24. Let $(X, \Delta)$ be a pair as in (6) and $f : X \to S$ a proper morphism. We say that $(X, \Delta)$ is an

$$\left.\begin{array}{l} f\text{-weak canonical} \\ f\text{-canonical} \\ f\text{-minimal} \end{array}\right\} \text{ model if } (X,\Delta) \text{ is } \left\{\begin{array}{c} \text{lc} \\ \text{lc} \\ \text{dlt} \end{array}\right\} \text{ and } K_X+\Delta \text{ is } \left\{\begin{array}{c} f\text{-nef.} \\ f\text{-ample.} \\ f\text{-nef.} \end{array}\right.$$

WARNING 25. Note that a canonical model $(X, \Delta)$ has *log* canonical singularities, not necessarily canonical singularities. This, by now entrenched, unfortunate terminology is a result of an incomplete shift. Originally everything was defined only for $\Delta = 0$. When $\Delta$ was introduced, its presence was indicated by putting "log" in front of adjectives. Later, when the use of $\Delta$ became pervasive, people started dropping the prefix "log". This is usually not a problem. For instance, the canonical ring $R(X, K_X)$ is just the $\Delta = 0$ special case of the log canonical ring $R(X, K_X + \Delta)$.

However, canonical singularities are not the $\Delta = 0$ special cases of log canonical singularities.

DEFINITION 26. Let $(X, \Delta)$ be a pair as in (5) such that $X$ is normal and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : X \to S$ be a proper morphism. A pair $(X^w, \Delta^w)$ sitting in a diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^w \\ f & \searrow \quad \swarrow & f^w \\ & S & \end{array}$$

is called a *weak canonical model* of $(X, \Delta)$ over $S$ if

(1) $f^w$ is proper,
(2) $\phi$ is a birational contraction, that is, $\phi^{-1}$ has no exceptional divisors,
(3) $\Delta^w = \phi_*\Delta$,
(4) $K_{X^w} + \Delta^w$ is $\mathbb{Q}$-Cartier and $f^w$-nef, and
(5) $a(E, X, \Delta) \le a(E, X^w, \Delta^w)$ for every $\phi$-exceptional divisor $E \subset X$.

As in (7), write

$$K_X + \phi_*^{-1}\Delta^w \sim_{\mathbb{Q}} \phi^*\bigl(K_{X^w} + \Delta^w\bigr) + \textstyle\sum_i a(E_i, X^w, \Delta^w)E_i, \tag{26.5.$a$}$$

where the $E_i$ are $\phi$-exceptional. Note that by (3), $\Delta = \phi_*^{-1}\Delta^w + \Delta^{ex}$ where $\Delta^{ex}$ is the $\phi$-exceptional part of $\Delta$. We can thus rewrite the above equation as

$$K_X + \Delta - \phi^*\bigl(K_{X^w} + \Delta^w\bigr) \sim_{\mathbb{Q}} \Delta^{ex} + \textstyle\sum_i a(E_i, X^w, \Delta^w)E_i. \tag{26.5.$b$}$$

We can thus restate (5) as

(5') $\Delta^{ex} + \sum_i a(E_i, X^w, \Delta^w)E_i$ is effective.

A weak canonical model $(X^m, \Delta^m) = (X^w, \Delta^w)$ is called a *minimal model* of $(X, \Delta)$ over $S$ if in addition to (1–4), we have

($5^m$) $a(E, X, \Delta) < a(E, X^m, \Delta^m)$ for every $\phi$-exceptional divisor $E \subset X$. Equivalently, if $\Delta^{ex} + \sum_i a(E_i, X^w, \Delta^w)E_i$ is effective and its support contains all $\phi$-exceptional divisor.

A weak canonical model $(X^c, \Delta^c) = (X^w, \Delta^w)$ is called a *canonical model* of $(X, \Delta)$ over $S$ if, in addition to (1–3) and (5) we have

($4^c$) $K_{X^c} + \Delta^c$ is $\mathbb{Q}$-Cartier and $f^c$-ample.

WARNING 27 (Pull-back by rational maps). If $g : X \dashrightarrow Y$ is a dominant rational map then one can define the pull back maps $g^* : \mathrm{CDiv}(Y) \to \mathrm{WDiv}(X)$ and $g^* : \mathrm{Pic}(Y) \to \mathrm{Cl}(X)$. Note, however, that if $h : Y \dashrightarrow Z$ is another dominant rational map then usually $(h \circ g)^* \neq g^* \circ h^*$.

A simple example is the following. Let $Y = \mathbb{P}^2$, $h : Y \dashrightarrow \mathbb{P}^1$ the projection from a point $y \in Y$ and $g : X := B_yY \to Y$ the blow up with exceptional curve $E$.

Then $h \circ g : B_yY \to \mathbb{P}^1$ is a morphism.

Let $D \subset \mathbb{P}^1$ be any effective divisor. Then $(h \circ g)^*D$ consists of the corresponding fibers of $B_yY \to \mathbb{P}^1$ and it never contains $E$. By contrast, $h^*D$ consists of a bunch of lines through $y$ and so $g^*(h^*D)$ contains $E$ with multiplicity $\deg D$.

(27.1) Note, however, that if $h$ is a morphism, then $(h \circ g)^* = g^* \circ h^*$.

Next we investigate when different choices of $\Delta$ lead to the same models.

PROPOSITION 28. *Let $\pi : X' \to X$ be a proper birational morphism of normal varieties. Let $\Delta$ and $\Delta'$ be $\mathbb{Q}$-divisors on $X$ and $X'$ such that $\pi_*\Delta' = \Delta$. Assume that*

(1) $a(E, X', \Delta') \leq a(E, X, \Delta)$ *for every $\pi$-exceptional divisor $E \subset X'$.*

*Then*

(2) *Every weak minimal model of $(X, \Delta)$ is also a weak minimal model of $(X', \Delta')$.*
(3) *Every canonical model of $(X', \Delta')$ is also a canonical model of $(X, \Delta)$.*
(4) *If all inequalities are strict in (1) then every weak minimal model of $(X', \Delta')$ is also a weak minimal model of $(X, \Delta)$.*

Proof. Let $(X^w, \Delta^w)$ be a weak minimal model of $(X, \Delta)$. If $E$ is any divisor on $X'$, then $a(E, X', \Delta') \leq a(E, X, \Delta)$ (and equality holds of $E$ is not $\pi$-exceptional). Thus, by (30.1), $a(E, X', \Delta') \leq a(E, X, \Delta) \leq a(E, X^w, \Delta^w)$. The other assumptions in (26) hold automatically, hence $(X^w, \Delta^w)$ is also a weak minimal model of $(X', \Delta')$.

Conversely, let $(X'^w, \Delta'^w)$ be a weak minimal model of $(X', \Delta')$ and $\phi' : X' \dashrightarrow X'^w$ the corresponding birational map. set $\phi := \phi' \circ \pi^{-1} : X \dashrightarrow X'^w$. Note that $\phi$ is a contraction iff the following assumption is satisfied.

(5) Every $\pi$-exceptional divisor $E \subset X'$ is contracted by $\phi'$.

If this holds, then $\Delta'^w = \phi'_*\Delta' = \phi_*\Delta$, thus the assumptions (26.1–4) are satisfied. Furthermore, if $F \subset X$ is $\phi$-exceptional, then $a(E, X, \Delta) = a(E, X', \Delta') \leq a(E, X'^w, \Delta'^w)$, thus (26.5) also holds and so $(X'^w, \Delta'^w)$ is a weak minimal model of $(X, \Delta)$.

Thus it remains to prove that (3) and (4) both imply (5). As in (26.5.b) we have

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} \phi'^*\big(K_{X'^w} + \Delta'^w\big) + F_1,$$

where $F_1$ is effective and supported on the $\phi'$-exceptional locus. By our assumptions

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} \pi^*(K_X + \Delta) + F_2,$$

where $F_2$ is effective and supported on the $\pi$-exceptional locus. Subtracting these two from each other, we obtain that

$$F_1 - F_2 \sim_{\mathbb{Q}} \pi^*(K_X + \Delta) - \phi'^*\big(K_{X'^w} + \Delta'^w\big).$$

Note that $\pi_*(F_1 - F_2) = \pi_*F_1$ is effective and $F_1 - F_2$ is $\pi$-nef since $K_{X'^w} + \Delta'^w$ is $f'^w$-nef. Thus, by (31), $F_1 - F_2$ is effective.

Assume now that (4) holds. Then the support of $F_2$ contains all $\pi$-exceptional divisor. These are the all contained in $\operatorname{Supp} F_1 \subset \operatorname{Supp} \operatorname{Ex} \phi'$, proving (5).

Finally, in order to prove (3), assume to the contrary that there is a $\pi$-exceptional divisor $E$ that is not $\phi'$-exceptional. Then $E$ appears in $F_1$ with coeffficient 0 and in $F_2$ with coeffficient $\geq 0$. Since $F_1 - F_2$ is effective, this is only possible if $E$ appears in $F_2$ with coeffficient 0 and them, by (31.2), $\phi'^*\big(K_{X'^w} + \Delta'^w\big)|_E$ is numerically $\pi$-trivial. Since $K_{X'^w} + \Delta'^w$ is $f'^w$-ample, this implies that $E$ is $\phi'$-exceptional. Thus (3) implies (5) and we are done. $\square$

LEMMA 29. *Consider a diagram as in (26).*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^w \\ f & \searrow \quad \swarrow & f^w \\ & S & \end{array}$$

*Let $\Delta_1, \Delta_2$ be $\mathbb{Q}$-divisors on $X$ such that $\Delta_1 \sim_{\mathbb{Q}} \Delta_2$. Assume that $\big(X^w, \phi_*\Delta_1\big)$ is an $f$-weak canonical (resp. $f$-canonical, $F$-minimal) model of $(X, \Delta_1)$.*

*Then $\big(X^w, \phi_*\Delta_2\big)$ is also an $f$-weak canonical (resp. $f$-canonical, $f$-minimal) model of $(X, \Delta_2)$.*

Proof. By assumption there is an $m \geq 1$ and a rational function $h$ on $X$ such that $\Delta_1 - \Delta_2 = \frac{1}{m}(h)_X$ where $(h)_X$ denotes the divisor of $h$ on $X$. Thus

$$\phi_*\Delta_1 - \phi_*\Delta_2 = \tfrac{1}{m}(h)_{X^w}.$$

In particular, $K_{X^w} + \phi_*\Delta_2$ is $\mathbb{Q}$-Cartier (resp. $f$-nef or $f$-ample) iff $K_{X^w} + \phi_*\Delta_2$ is. As in (26.5.b), write

$$K_X + \Delta_i - \phi^*\big(K_{X^w} + \phi_*\Delta_i\big) \sim_{\mathbb{Q}} \Delta_i^{ex} + \textstyle\sum_j a_{ij}E_j. \tag{29.1.$i$}$$

By definition, $\big(X^w, \phi_*\Delta_i\big)$ is an $f$-weak canonical (resp. $f$-minimal) model of $(X, \Delta_i)$ iff the right hand side of (29.1.i) is effective (resp. effective and with support $\operatorname{Ex}(\phi)$).

Subtracting the equations (29.1.i) from each other we obtain that

$$\Delta_1^{ex} + \textstyle\sum_j a_{1j}E_j \sim_{\mathbb{Q}} \Delta_2^{ex} + \textstyle\sum_j a_{2j}E_j.$$

Thus, a multiple of their difference is a function divisor $(g)_X$. Since these divisors are $\phi$-exceptional, we see that $(g)_{X^w} = 0$. Thus $g$ is a regular function on $X^w$, hence on $S$ and also on $X$. Therefore

$$\Delta_1^{ex} + \textstyle\sum_j a_{1j}E_j = \Delta_2^{ex} + \textstyle\sum_j a_{2j}E_j. \tag{29.2}$$

This completes the proof. $\square$

Next we consider results that connect properties of $(X, \Delta)$ and of its weak canonical models.

PROPOSITION 30. *Let $\phi : (X, \Delta) \dashrightarrow (X^w, \Delta^w)$ be a weak canonical model. Then*

(1) *$a(E, X^w, \Delta^w) \geq a(E, X, \Delta)$ for every divisor $E$ and the inequality is strict if $(X^w, \Delta^w)$ is a weak minimal model and $\phi$ is not a local isomorphism at the generic point of* $\operatorname{center}_X E$,
(2) $\operatorname{totaldiscrep}(X^w, \Delta^w) \geq \operatorname{totaldiscrep}(X, \Delta)$ *and*
(3) $\operatorname{discrep}(X^w, \Delta^w) \geq \min\big\{\operatorname{discrep}(X, \Delta), a(E_i, X, \Delta) : i \in I\big\}$ *where the $\{E_i : i \in I\}$ are the $\phi$-exceptional divisors.*
(4) *If $(X, \Delta)$ is lc (resp. klt) then so is $(X^w, \Delta^w)$.*

Proof. Fix $E$ and consider any diagram

$$\begin{array}{ccccc} & & Y & & \\ & g & \swarrow \quad \searrow & h & \\ & X & \stackrel{\phi}{\dashrightarrow} & X^w & \\ & f & \searrow \quad \swarrow & f^w & \\ & & S & & \end{array}$$

where $(X^w, \Delta^w)$ is a weak canonical model and $\operatorname{center}_Y E$ is a divisor. Write

$$\begin{aligned} K_Y &\sim_{\mathbb{Q}} g^*(K_X + \Delta) + E_1, \quad \text{and} \\ K_Y &\sim_{\mathbb{Q}} h^*(K_{X^w} + \Delta^w) + E_2. \end{aligned}$$

Notice that $a(E, X^w, \Delta^w) - a(E, X, \Delta)$ is the coefficient of $E$ in $E_2 - E_1$. Set

$$B := g^*(K_X + \Delta) - h^*(K_{X^w} + \Delta^w) \sim_{\mathbb{Q}} E_2 - E_1.$$

Then $-B$ is $g$-nef and $g_*B = g_*(E_2 - E_1)$ is effective by (26.5). Thus $E_2 - E_1$ is effective by (31).

If $(X^w, \Delta^w)$ is a weak minimal model and $\phi$ is not a local isomorphism at the generic point of $\operatorname{center}_X E$ then $\operatorname{center}_X E \subset \operatorname{Supp} g_*(E_2 - E_1)$, thus $E \subset \operatorname{Supp}(E_2 - E_1)$ by (31.2).

Since $\operatorname{discrep}(Y, D)$ is the minimum of all discrepancies of divisors which are exceptional over $Y$, we see that the $\phi$-exceptional $E_i$ are taken into account when computing $\operatorname{discrep}(X^w, \Delta^w)$ but not in the computation of $\operatorname{discrep}(X, \Delta)$. $\square$

The next lemma is useful in many situations.

Lemma 31. [**KM98**, 3.39] *Let $h : Z \to Y$ be a proper birational morphism between normal schemes. Let $-B$ be an $h$-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Z$. Then*

1. *$B$ is effective iff $h_*B$ is.*
2. *Assume that $B$ is effective. Then for every $y \in Y$, either $h^{-1}(y) \subset \operatorname{Supp} B$ or $h^{-1}(y) \cap \operatorname{Supp} B = \emptyset$.*

*Thus if $B$ is also $h$-exceptional then $B$ is numerically $h$-trivial iff $B = 0$.* $\square$

Next we consider a result that makes it easy to decide when two pairs have the same canonical rings.

Theorem 32. *Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ be proper morphisms of normal schemes and $\phi : X_1 \dashrightarrow X_2$ a birational map such that $f_1 = f_2 \circ \phi$. Let $\Delta_1$ and $\Delta_2$ be $\mathbb{Q}$-divisors such that $K_{X_1} + \Delta_1$ and $K_{X_2} + \Delta_2$ are $\mathbb{Q}$-Cartier. Then*

$$f_{1*}\mathcal{O}_{X_1}(mK_{X_1} + \lfloor m\Delta_1 \rfloor) = f_{2*}\mathcal{O}_{X_2}(mK_{X_2} + \lfloor m\Delta_2 \rfloor) \quad \text{for } m \geq 0$$

*if the following conditions hold:*

1. *$a(E, X_1, \Delta_1) = a(E, X_2, \Delta_2)$ if $\phi$ is a local isomorphism at the generic point of $E$,*
2. *$a(E, X_1, \Delta_1) \leq a(E, X_2, \Delta_2)$ if $E \subset X_1$ is $\phi$-exceptional, and*
3. *$a(E, X_1, \Delta_1) \geq a(E, X_2, \Delta_2)$ if $E \subset X_2$ is $\phi^{-1}$-exceptional.*

Proof. Let $Y$ be the normalization of the main component of $X_1 \times_S X_2$ and $g_i : Y \to X_i$ the projections. We can write

$$\begin{aligned} K_Y &\sim_{\mathbb{Q}} g_1^*(K_{X_1} + \Delta_1) + \textstyle\sum_E a(E, X_1, \Delta_1)E, \quad \text{and} \\ K_Y &\sim_{\mathbb{Q}} g_1^*(K_{X_2} + \Delta_2) + \textstyle\sum_E a(E, X_2, \Delta_2)E. \end{aligned}$$

Set $b(E) := \max\{-a(E, X_1, \Delta_1), -a(E, X_2, \Delta_2)\}$. By (20), it is sufficient to prove that

$$f_{i*}\mathcal{O}_{X_i}(mK_{X_i} + \lfloor m\Delta_i \rfloor) = (f_i \circ g_i)_*\mathcal{O}_Y(mK_Y + \textstyle\sum_E \lfloor mb(E) \rfloor E) \tag{32.4}$$

holds for $i = 1, 2$ and any sufficiently divisible $m > 0$. Observe that

$$\begin{aligned} K_Y + \textstyle\sum_E b(E)E \quad &\sim_{\mathbb{Q}} \quad g_1^*(K_{X_1} + \Delta_1) + \textstyle\sum_E \big(b(E) + a(E, X_1, \Delta_1)\big)E, \quad \text{and} \\ K_Y + \textstyle\sum_E b(E)E \quad &\sim_{\mathbb{Q}} \quad g_1^*(K_{X_2} + \Delta_2) + \textstyle\sum_E \big(b(E) + a(E, X_2, \Delta_2)\big)E. \end{aligned}$$

Note that $\sum_E \big(b(E) + a(E, X_i, \Delta_i)\big)E$ is effective by the definition of $b(E)$. Furthermore, if $E$ is not $g_1$-exceptional then either $\phi$ is a local isomorphism at the generic point of $E$ (and thus $b(E) = -a(E, X_1, \Delta_1) = -a(E, X_2, \Delta_2)$) or $E \subset X_1$ is $\phi$-exceptional (and thus $b(E) = -a(E, X_1, \Delta_1) \geq -a(E, X_2, \Delta_2)$). A similar argument applies to $g_2$. Therefore $\sum_E \big(b(E) + a(E, X_i, \Delta_i)\big)E$ is effective and $g_i$-exceptional for $i = 1, 2$, Thus, for sufficiently divisible $m > 0$,

$$\begin{aligned} &(f_i \circ g_i)_*\mathcal{O}_Y(mK_Y + \textstyle\sum_E mb(E)E) \\ &\quad = f_{i*}g_{i*}\mathcal{O}_Y\big(g_1^*(mK_{X_1} + m\Delta_1) + \textstyle\sum_E \big(mb(E) + ma(E, X_1, \Delta_1)\big)E\big) \\ &\quad = f_{i*}\mathcal{O}_{X_i}(mK_{X_i} + m\Delta_i), \end{aligned}$$

giving (32.4). □

Applying (26) to $X_1 = X$ and $X_2 = X^w$ gives the following.

Corollary 33. *Let $(X, \Delta)$ be a pair as in (26) and $f^w : (X^w, \Delta^w)$ a weak canonical model. Then $f_*\mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor) = f^w_*\mathcal{O}_{X^w}(mK_{X^w} + \lfloor m\Delta^w \rfloor)$ for every $m \geq 0$.* □

The following is a straightforward generalization of (22).

Theorem 34. *Let $(X, \Delta)$ be a pair as in (5) with $X$ normal and $f : X \to S$ a proper morphism. Asume that its (relative) canonical ring*

$$R(X/S, K_X + \Delta) := \sum_{m \geq 0} f_*\mathcal{O}_X\big(mK_X + \lfloor m\Delta \rfloor\big)$$

*is a finitely generated sheaf of $\mathcal{O}_S$-algebras and $K_X + \Delta$ is big on the generic fiber of $f$. Then*

(1) *$X^{\mathrm{can}} := \mathrm{Spec}_S \sum_{m \geq 0} f_*\mathcal{O}_X\big(mK_X + \lfloor m\Delta \rfloor\big)$ is normal, projective over $S$ and there is a natural birational map $\phi : X \dashrightarrow X^{\mathrm{can}}$,*
(2) *The class $K_{X^{\mathrm{can}}} + \Delta^{\mathrm{can}}$ is $\mathbb{Q}$-Cartier and ample over $S$ where $\Delta^{\mathrm{can}} := \phi_*\Delta$.*
(3) *$(X^{\mathrm{can}}, \Delta^{\mathrm{can}})$ is the unique canonical model of $(X, \Delta)$.*
(4) *Push-forward by $\phi$ gives an isomorphism*

$$\sum_{m \geq 0} f_*\mathcal{O}_X\big(mK_X + \lfloor m\Delta \rfloor\big) \cong \sum_{m \geq 0} f^{\mathrm{can}}_*\mathcal{O}_{X^{\mathrm{can}}}\big(mK_{X^{\mathrm{can}}} + \lfloor m\Delta^{\mathrm{can}} \rfloor\big).$$

(5) *If $(X, \Delta)$ is log canonical then so is $(X^{\mathrm{can}}, \Delta^{\mathrm{can}})$.* □

Proposition 35. *Let $(X, \Delta)$ be a pair as in (26). Then any two minimal models of $(X, \Delta)$ are isomorphic in codimension one.*

Proof. Let $\phi_i : X \dashrightarrow X_i^m$ be two minimal models. We need to show that $\phi_1$ and $\phi_2$ have the same exceptional divisors. We can choose $g : Y \to X$ such that $h_i : Y \to X_i^m$ are both morphisms. We obtain that

$$K_Y + \Delta_Y \sim_{\mathbb{Q}} h_i^*(K_{X_i^m} + \Delta_i^m) + Z_i,$$

where $Z_i$ is effective and $\operatorname{Supp} Z_i$ contains all exceptional divisors of $\phi_i$ by ($26.5^m$). Combining the two formulas we get

$$h_1^*(K_{X_1^m} + \Delta_1^m) - h_2^*(K_{X_2^m} + \Delta_2^m) \sim_{\mathbb{Q}} Z_2 - Z_1.$$

Applying (31) to $h_1 : Y \to X_1^m$ (resp. $h_2 : Y \to X_2^m$) we obtain that $Z_2 - Z_1$ (resp. $Z_1 - Z_2$) is effective. Thus $Z_1 = Z_2$ and so $\phi_1$ and $\phi_2$ have the same exceptional divisors. □

**Existence of Canonical Models.**

Let us start with a very general form of the minimal model conjecture.

Conjecture 36 (Minimal Model Conjecture). *Let $f : X \to S$ be a proper morphism between normal schemes or algebraic spaces with generic fiber $X_{gen}$ (over $f(X)$). Let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $(X, \Delta)$ is lc. Then*

(1) *$(X, \Delta)$ has a minimal model $(X^m, \Delta^m)$ iff the restriction of $K_X + \Delta$ to the generic fiber $X_{gen}$ is pseudo effective (that is, it is numerically equivalent to a limit of effective $\mathbb{Q}$-divisors). Furthermore, if $X$ is $\mathbb{Q}$-factorial then one can choose $X^m$ to be $\mathbb{Q}$-factorial.*
(2) *$(X, \Delta)$ has a canonical model iff the restriction of $K_X + \Delta$ to $X_{gen}$ is big.*

37 (Known special cases). The minimal model conjecture (36) is known in many instances.

*Surfaces 37.1.* The full conjecture is known if $\dim X \leq 2$. If $X$ is smooth over a field $k$, this is the classical theory of minimal models, see [**BPVdV84**, Sec.III.4]. The general case can be established along the same lines using resolution of singularities [**Sha66**] and general contractibility criteria for curves on surfaces [**Lip69**].

*Threefolds 37.2.* The conjecture is known if $\dim X \leq 3$ and $S$ is essentially of finite type over a field of characteristic 0. The case when $X$ is terminal and $\Delta = 0$ is due to [**Mor88**]. The first part in the klt case was proved by [**Sho92**]. A different proof and a seminar-style work out of both parts in the lc case is in [**K$^+$92**]. Very little is known in positive characteristic [**Kol91**]. Some cases when $\dim S = 1$ are proved in [**Kaw94, Kaw99**].

*4-folds 37.3.* The conjecture is known if $\dim X \leq 4$, $S$ is essentially of finite type over a field of characteristic 0 and $K_{X_{gen}} + \Delta_{gen}$ is effective. The first part is due to [**Sho06**] and the second part follows from this and [**Fuj00**].

*5-folds 37.4.* The first part of the conjecture is known if $\dim X \leq 5$, $S$ is essentially of finite type over a field of characteristic 0 and $K_{X_{gen}} + \Delta_{gen}$ is effective [**Bir07**].

*n-folds 37.5.* The conjecture is known if $S$ is essentially of finite type over a field of characteristic 0, $(X, \Delta)$ is klt and $K_{X_{gen}} + \Delta_{gen}$ is big [**BCHM06**]. When $S = \operatorname{Spec} \mathbb{C}$, $X$ is canonical and $\Delta = 0$, the existence of the canonical model is also in [**Siu08**].

38 (MMP and Moduli problems). In our approach to the moduli of varieties of general type, we need (36) when $(X, \Delta)$ is dlt and $S$ is either the spectrum of a field or of a Dedekind ring. While it would be convenient to allow arbitrary Dedekind rings, it is enough to consider Dedekind rings that are localizations of finite-type algebras over a field or over $\mathbb{Z}$.

Currently these cases are known only if

(1) $\dim X \le 4$, and
(2) $S$ is over a field of characteristic 0.

MMP is a very active research area, and the first of these restrictions may be removed in the near future.

By contrast, very little work has been done about the cases when $S$ has positive or mixed characteristic.

39 (Some dlt cases). There are some cases when the existence of dlt minimal models can be reduced to the klt case using (29).

(39.1) Assume that $(X, \Delta = \sum_{i\ge 0} d_i D_i)$ is dlt and $D_0$ is ample. Then, for $0 < \epsilon_i \ll \epsilon_0 \ll 1$, $\sum_i \epsilon_i D_i$ is still ample, hence linearly equivalent to $\frac{1}{m}H$ where $H$ is a general member of a very ample linear system $\bigl|m\bigl(\sum_i \epsilon_i D_i\bigr)\bigr|$. Thus

$$\Delta_\epsilon := \tfrac{1}{m}H + \sum_{i\ge 0}(d_i - \epsilon_i)D_i \sim_{\mathbb{Q}} \Delta$$

and $(X, \Delta_\epsilon)$ is klt. Thus by (29), a weak canonical model $\phi : (X, \Delta_\epsilon) \dashrightarrow (X^w, \Delta_\epsilon^w)$ is also a weak canonical model of $(X, \Delta)$.

(39.2) Let $(X, \Delta)$ be an arbitrary dlt pair with $X$ projective. Pick any ample divisor $H$ and apply the above arguments to $(X, \Delta + \epsilon H)$ for $\epsilon > 0$. Conjecturally, for $0 < \epsilon \ll 1$, a weak minimal model of $(X, \Delta + \epsilon H)$ is also a weak minimal model of $(X, \Delta)$, but this is not known. However, as we let $\epsilon \to 0$, we obtain a sequence of minimal models which function as better and better approximations of a weak minimal model of $(X, \Delta)$ (whose existence is not known).

The following method gives us great flexibility in choosing a birational model.

40 (Picking birational models). Let $(X, \Delta)$ be a pair such that $X$ is quasi-projective, $\Delta = \sum_{j\in J} d_j D_j$ where $0 \le d_j \le 1$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. (We do not assume that $(X, \Delta)$ is lc.) Let $f : Y \to (X, \Delta)$ be a log resolution that is a composite of blow-ups of centers of codimension at least 2 and $\{E_i : i \in I\}$ the $f$-exceptional divisors. Pick a rational number $0 \le c_i \le 1$ for every $i$.

Then there is a proper birational morphism $g : X^{\mathrm{mc}} \to X$ with exceptional divisors $\{E_i : i \in I^{\mathrm{mc}}\}$ for some $I^{\mathrm{mc}} \subset I$ such that

(1) $X^{\mathrm{mc}}$ is a minimal model for a suitable $\bigl(Y, \sum c_i' E_i\bigr)$ for some $c_i' < c_i$,
(2) $\bigl(X^{\mathrm{mc}}, g_*^{-1}\Delta + \sum_{i\in I} c_i E_i\bigr)$ is dlt,
(3) $a(E_i, X, \Delta) \le -c_i$ for every $i \in I^{\mathrm{mc}}$,
(4) if $a(E_i, X, \Delta) \le -c_i$ for some $i \in I$ then $I^{\mathrm{mc}} \ne \emptyset$,
(5) $X^{\mathrm{m}}$ is $\mathbb{Q}$-factorial and
(6) there is a divisor $B = \sum b_i E_i$ such that $-B$ is $g$-nef and $b_i > 0$ for every $i \in I^{\mathrm{mc}}$.

In applications the key question is the choice of the coefficients $c_i$. Varying the $c_i$ gives different models $X^{\mathrm{mc}}$. Let us see how this works in some examples.

COROLLARY 41 (Dlt models). *Let $(X, \Delta)$ be as in (40). Then there is a proper birational morphism $g : X^{\mathrm{m}} \to X$ with exceptional divisors $E_i$ such that*

(1) $\bigl(X^{\mathrm{m}}, g_*^{-1}\Delta + \sum E_i\bigr)$ *is dlt.*
(2) $a(E_i, X, \Delta) \le -1$ *for every $i$.*
(3) $X^{\mathrm{m}}$ *is $\mathbb{Q}$-factorial.*

(4) *If* $(X, \Delta)$ *is lc then* $K_{X^{\mathrm{m}}} + g_*^{-1}\Delta + \sum E_i \sim_{\mathbb{Q}} g^*(K_X + \Delta)$.
(5) *If* $(X, \Delta)$ *is not lc then there is an exceptional divisor* $E_i$ *such that* $a(E_i, X, \Delta) < -1$.

Proof. Choose $c_i = 1$ for every $i$ and apply (40).

Only the last statement needs proof. If there is no such divisor, then $a(E_i, X, \Delta) = -1$ for every $i$, hence $K_{X^{\mathrm{m}}} + g_*^{-1}\Delta + \sum E_i \sim_{\mathbb{Q}} g^*(K_X + \Delta)$. By (1), $\big(X^{\mathrm{m}}, g_*^{-1}\Delta + \sum E_i\big)$ is dlt, hence lc, and by (12) so is $(X, \Delta)$; a contradiction. □

COROLLARY 42 ($\mathbb{Q}$-factorial models). *Let* $(X, \Delta)$ *be dlt. Then there is a proper birational morphism* $g : X^{\mathrm{qf}} \to X$ *such that*

(1) $X^{\mathrm{qf}}$ *is* $\mathbb{Q}$*-factorial,*
(2) $g$ *is small, that is, without* $g$*-exceptional divisors and*
(3) $\big(X^{\mathrm{qf}}, g_*^{-1}\Delta\big)$ *is dlt.*

Proof. By (**??**) we can take a log resolution $f : Y \to (X, \Delta)$ such that every $f$-exceptional divisor has discrepancy $> -1$. Pick all $c_i = 1$ and apply (40) to get $g : X^{\mathrm{qf}} \to X$. By (40.3), all the $f$-exceptional divisors get killed, hence $g : X^{\mathrm{qf}} \to X$ is small. □

COROLLARY 43 (Extracting one divisor). *Let* $(X, \Delta)$ *be klt and* $E$ *an exceptional divisor over* $X$ *such that* $a(E, X, \Delta) \le 0$. *Then there is a proper birational morphism* $g : X^E \to X$ *such that*

(1) $X^E$ *is* $\mathbb{Q}$*-factorial and*
(2) $E$ *is the sole exceptional divisor of* $g$.

Proof. Pick any log resolution $Y \to X$ such that $E = E_0$ is a divisor on $Y$. Choose all $c_i = 1$ except $c_0 = 0$ and apply (40). By (40.3–4) the only divisor that survives is $E$. □

44 (Proof of (40)). We claim that there is an effective $f$-exceptional divisor $C$ such that $-C$ is $f$-ample. For one blow-up, the exceptional divisor is relatively anti-ample. The rest follows by repeatedly appying [**Laz04**, 1.7.10].

If $H$ is sufficiently ample on $X$ then $H' := f^*H - C$ is ample on $Y$. Write $C = \sum \gamma_i E_i$ and set $\gamma_i = 0$ and $c_i = -a_i(E_i, X, \Delta)$ for the non-exceptional $E_i$. Note that

$$K_Y + \sum c_i E_i + \eta H' = K_Y + \sum \big(c_i - \eta\gamma_i\big) E_i + \eta f^* H.$$

By (39.1) there is a $\mathbb{Q}$-factorial minimal model $g : X^{\mathrm{mc}} \to X$ for $(Y, \sum c_i E_i + \eta H')$. In particular, $(X^{\mathrm{mc}}, \sum c_i E_i^m)$ is dlt. By (29), $g : X^{\mathrm{mc}} \to X$ is also a minimal model for $(Y, \sum\big(c_i - \eta\gamma_i\big)E_i)$. In particular, $K_{X^{\mathrm{mc}}} + \sum(c_i - \eta\gamma_i)E_i^m$ if $g$-nef. By the definition of discrepancies, $K_{X^{\mathrm{mc}}} - \sum a(E_i, X, \Delta)E_i^m$ is $g$-trivial. Hence their difference

$$-B := \sum \big(c_i + a(E_i, X, \Delta) - \eta\gamma_i\big) E_i^m$$

is $g$-nef. By our choice, $g_*B = 0$, hence $B$ is effective by (31.1). Thus $a(E_i, X, \Delta) \le -c_i + \eta\gamma_i$ if $E_i$ is a divisor on $X^{\mathrm{mc}}$. Since these $E_i$ are among the finitely many $f$-exceptional divisors, by choosing $0 < \eta \ll 1$, we get that in fact $a_i(E_i, X, \Delta) \le -c_i$.

Assume that $I^{\mathrm{mc}} = \emptyset$, that is, $g$ is small. Then $(X, \Delta)$ is a weak minimal model of $\big(Y, \sum(c_i - \eta\gamma_i)E_i\big)$. By (30) then $c_i - \eta\gamma_i > a(X, \Delta, E_i)$ for every $i \in I$, proving (40.4). □

## 2. Examples of log canonical singularities

The simplest examples of terminal, canonical etc. singularities are given by cones. (See (56) for our conventions on cones.) Cones are rather special, but illustrate many of the difficulties that appear when dealing with these singularities.

The next lemma follows directly from (62). In most of the subsequent examples we choose $X$ to be smooth. Then the assumptions that $X$ be terminal (resp. canonical, ...) are all satisfied.

Lemma 45. *Let $X$ be normal, projective, $H$ an ample Cartier divisior on $X$ and $C_a(X,H)$ the corresponding affine cone (56). Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ and $\Delta_{C_a(X,H)}$ the corresponding $\mathbb{Q}$-divisor on $C_a(X,H)$. Assume that $K_X+\Delta\sim_{\mathbb{Q}} r\cdot H$ for some $r\in\mathbb{Q}$. Then $\bigl(C_a(X,H),\Delta_{C_a(X,H)}\bigr)$ is*

1. *terminal iff $r<-1$ and $(X,\Delta)$ is terminal,*
2. *canonical iff $r\le -1$ and $(X,\Delta)$ is canonical,*
3. *klt iff $r<0$ (that is, $-(K_X+\Delta)$ is ample) and $(X,\Delta)$ is klt,*
4. *lc iff $r\le 0$ (that is, $-(K_X+\Delta)$ is nef) and $(X,\Delta)$ is lc.* □

Both of the next two results are proved by an easy case analysis.

Corollary 46. *Let $B$ be a smooth curve and $L$ an ample line bundle on $B$. Then $C_a(B,L)$ is*

1. *terminal iff $g(B)=0$ and $\deg L=1$ ($C_a(B,L)$ is an affine plane),*
2. *canonical iff $g(B)=0$ and $\deg L\le 2$ ($C_a(B,L)$ is an affine plane or a quadric cone),*
3. *log terminal iff $g(B)=0$ ($C_a(B,L)$ is a cone over a rational normal curve) and*
4. *log canonical iff $g(B)=1$ (or (3) holds).* □

The easiest higher dimensional case is the following:

Corollary 47. *Let $X\subset\mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of degrees $(d_1,\dots,d_m)$. Then $C(X)=C_a\bigl(X,\mathcal{O}_X(1)\bigr)$ and it is*

1. *terminal iff $\sum d_i<n$,*
2. *canonical iff $\sum d_i\le n$,*
3. *log canonical iff $\sum d_i\le n+1$.* □

In constructing a moduli theory for varieties of general type, one needs to pay special attention to two properties.

48 (Rational and Cohen-Macaulay singularities). Being *Cohen-Macaulay*, or *CM*, is a somewhat technical but very useful condition for schemes and coherent sheaves; see [**Mat86**, Sec.17], [**Har77**, pp.184–186]. Roughly speaking the basic cohomology theory of CM coherent sheaves works very much like the cohomology theory of locally free sheaves on a smooth variety. Indeed, let $X$ be a projective scheme of pure dimension $n$ over a field $k$ and $F$ a coherent sheaf on $X$. By the Noether normalization theorem, there is a finite morphism $\pi: X\to\mathbb{P}^n$. Then

- $H^i(X,F)=H^i(\mathbb{P}^n,\pi_*F)$ and
- $F$ is CM iff $\pi_*F$ is locally free.

One can use these to develop the duality theory for projective schemes, see [**KM98**, Sec.5.5].

Let $X$ be a normal variety over a field of characteristic 0. We say that $X$ has *rational* singularities iff $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$ for every resolution of singularities and $f : Y \to X$. It is actually enough to check this for one resolution, or even for one proper birational $f : Y \to X$ such that $Y$ has rational singularities. See [**KM98**, Sec.5.1] for an introduction to rational singularities and Section 4 for further results. (In positive characteristic one also needs to assume that $R^i f_*\omega_Y = 0$ for $i > 0$, which is automatic in characteristic 0; see [**KM98**, 2.68].)

Using the Leray spectral sequence $H^i(X, R^j f_*\mathcal{O}_Y) \Rightarrow H^{i+j}(Y, \mathcal{O}_Y)$ we obtain that $H^i(X, \mathcal{O}_X) = H^i(Y, \mathcal{O}_Y)$. More generally, for any vector bundle $E$ on $X$

$$H^i(X, E) = H^i(Y, f^*E).$$

Thus, many cohomology computations on the singular variety $X$ can be reduced to a computation on the smooth variety $Y$.

Rational implies CM, but this is not obvious; see [**KM98**, 5.10].

It is easy to decide when a cone considered in (45) is CM:

Corollary 49. *Let $X$ be a smooth, projective variety over $\mathbb{C}$ and $L$ an ample line bundle on $X$.*

1. *If $-K_X$ is ample (that is, $X$ is Fano) then $C_a(X, L)$ is CM and has rational singularities.*
2. *If $-K_X$ is nef (for instance, $X$ is Calabi-Yau), then*
   (a) *$C_a(X, L)$ is CM iff $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$, and*
   (b) *$C_a(X, L)$ has rational singularities iff $H^i(X, \mathcal{O}_X) = 0$ for $0 < i \leq \dim X$.*

Proof. By assumption $\omega_X \otimes L^{-m}$ is anti-ample for $m \geq 1$, and even for $m = 0$ if $-K_X$ is ample. Hence, by Kodaira vanishing and by Serre duality, $H^i(X, L^m) = 0$ for $i > 0$ and $m \geq 0$ in case (1) and $m > 0$ in case (2). Thus in (59) we only need the vanishing of $H^i(X, L^0) = H^i(X, \mathcal{O}_X)$ for $0 < i < \dim X$. This proves (1) and (2.a) while (2.b) follows from (61). □

The following examples are proved by a straightforward application of (49).

Example 50 (Log canonical but not CM). The following examples show that log canonical singularities need not be rational, not even CM. The claims follow directly from (49) and hold for every ample line bundle $L$.

1. A cone over an Abelian variety $A$ is CM iff $\dim A = 1$.
2. A cone over a K3 surface is CM but not rational.
3. A cone over an Enriques surface is CM and rational.
4. A cone over a smooth Calabi-Yau complete intersection is CM but not rational.

While not related to any of our questions, the following example shows that one of the standard constructions, the taking of the canonical cover, can also lead to more complicated singularities.

Example 51 (Canonical covers). Let $x \in X$ be a normal singularity over $\mathbb{C}$ and assume that $K_X$ is $\mathbb{Q}$-Cartier. Then there is a smallest $m > 0$ such that $mK_X$ is Cartier near $x$. By replacing $X$ with a smaller neighborhood of $x$, we may assume that $mK_X \sim 0$. Let $X^{sm}$ be the smooth locus of $X$. Thus $L := \mathcal{O}_{X^{sm}}(K_{X^{sm}})$ is a line bundle such that $L^m$ is the trivial line bundle on $X^0$. Thus $c_1(L) \in H^2(X^{sm}, \mathbb{Z})$

is $m$-torsion and so it corresponds to a degree $m$ étale cover $\pi^0 : \tilde{X}^0 \to X^0$ such that $(\pi^0)^*L$ is trivial. Extend $\pi^0$ to a ramified finite cover $\pi : \tilde{X} \to X$, called the *canonical cover* or *index 1 cover* of $X$. Then $K_{\tilde{X}} = \pi^* K_X$ is Cartier. For a purely algebraic construction see [**KM98**, 2.49].

For the singularities of the MMP, the canonical cover $\tilde{X}$ is usually simpler than the original singularity $X$. Nonetheless, here are 4 examples to show that the canonical cover $\tilde{X}$ can be cohomologically more complicated than $X$; see also [**Sin03**].

1. Let $S$ be a K3 surface with a fixed point free involution $\tau$. Thus $T := S/\tau$ is an Enriques surface. A cone $C_a(T)$ over the Enriques surface $T$ is CM and rational. By (62) the canonical class of $C_a(T)$ is not Cartier but $2K_{C_a(T)}$ is Cartier. Its canonical cover is a cone over the K3 surface $S$ which is CM but not rational.

2. Let $S$ be a K3 surface with a fixed point free involution $\tau$ as above. Then $\tau$ acts as multiplication by $-1$ on $H^2(S, \mathcal{O}_S)$. By the Küneth formula, $(\tau, \tau)$ acts as multiplication by $-1$ on $H^2(S \times S, \mathcal{O})$ and as multiplication by $1 = (-1) \cdot (-1)$ on $H^4(S \times S, \mathcal{O})$. Set $X := (S \times S)/(\tau, \tau)$. Then $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < 4$, hence a cone over $X$ is CM. Its canonical cover is a cone over $S \times S$ which is not CM.

3. Let $A$ be the Jacobian of the hyperelliptic curve $y^2 = x^5 - 1$ and $\tau$ the automorphism induced by $(x, y) \mapsto (\epsilon x, y)$ where $\epsilon$ is a 5th root of unity. The space of holomorphic 1-forms is spanned by $dx/y, xdx/y$. Thus $\tau$ acts on $H^1(A, \mathcal{O}_A)$ with eigenvalues $\epsilon, \epsilon^2$ and on $H^2(A, \mathcal{O}_A) = \wedge^2 H^1(A, \mathcal{O}_A)$ with eigenvalue $\epsilon^3$. Set $S := A/\langle\tau\rangle$. Then $H^i(S, \mathcal{O}_S) = 0$ for $i = 1, 2$, hence a cone over $S$ is a rational singularity. Its canonical cover is a cone over $A$, which is not CM.

4. Let $S_m$ be a K3 surface and $\sigma_m$ an automorphism of order $m$ whose fixed point set is finite and such that $\sigma_m$ acts as multiplication by $\epsilon_m$ on $H^2(S_m, \mathcal{O}_{S_m})$ where $\epsilon_m$ is a primitive $m$th root of unity. (Such $S_m$ and $\sigma_m$ exists for $m \in \{5, 7, 11, 17, 19\}$ by [**Kon92**].) By the Küneth formula, $(\sigma_m, \sigma_m)$ acts as multiplication by $\epsilon_m$ on $H^2(S_m \times S_m, \mathcal{O})$ and as multiplication by $\sigma_m^2$ on $H^4(S_m \times S_m, \mathcal{O})$. Set $X_m := (S_m \times S_m)/(\sigma_m, \sigma_m)$. Then $h^i(X_m, \mathcal{O}_{X_m}) = 0$ for $0 < i \leq 4$, hence a cone over $X_m$ is rational.

By (62) the canonical class of $C_a(X_m)$ is not Cartier but $mK_{C_a(X_m)}$ is Cartier. Its canonical cover is a cone over $S_m \times S_m$ which is not even CM.

**Perturbing the boundary.**

Starting with a lc pair $(X, \Delta)$, it is frequently very useful to perturb $\Delta$ in order to increase the disrepancy of the pair $(X, \Delta)$. By (10), we need to decrease $\Delta$ if we want to increase the disrepancy. This is not possible if $\Delta = 0$ to start with, but for most lc pairs $(X, \Delta)$ such that $(X, 0)$ is klt outside $\operatorname{Supp} \Delta$, there is a divisor $0 \leq \Delta' \leq \Delta$ such that $(X, \Delta')$ is klt.

However, the following series of examples show that this is not always the case.

PROPOSITION 52. *Let $X$ be proper and $E \subset X$ an irreducible $\mathbb{Q}$-Cartier divisor such that $K_X + cE \sim_{\mathbb{Q}} 0$ for some $0 < c \leq 1$ and $E|_E$ is not $\mathbb{Q}$-linearly equivalent to an effective divisor.*

*Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta \sim_{\mathbb{Q}} 0$. Then $\Delta = cE$.*

Proof. Write $\Delta = (c - \gamma)E + \Delta'$ where $\Delta'$ is effective, its suport does not contain $E$ and $\gamma \in \mathbb{Q}$. Then $\Delta' \sim_{\mathbb{Q}} \gamma E$ which shows that $\gamma \geq 0$. Restricting to $E$ we get $\gamma\big(E|_E\big) \sim_{\mathbb{Q}} \Delta'|_E$. This is a contradiction, unless $\gamma = 0$ and $\Delta = cE$. $\square$

Corollary 53. *Let $(X, cE)$ be as in (52). Let $H$ be an ample Cartier divisor on $X$ and $Y := C_a(X, H)$ the affine cone over $X$ with vertex $v$. Let $\Delta_Y$ be an effective $\mathbb{Q}$-divisor on $Y$ such that $(Y, \Delta_Y)$ is lc. Then $v$ is an lc center of $(Y, \Delta_Y)$.*

Note that, in contrast with (52), we do not claim that $\Delta_Y = cC_a(E, H|_E)$ and this is not at all true. In fact, if $X$ is smooth then $C_a(E, H|_E)$ determines an (infinite dimensional) linear system whose only base point is at $v$. If $D$ is a general member, then $(Y, cD)$ is lc. Or we can take $r$ general members $D_1, \dots, D_r$ and then $\bigl(Y, \frac{c}{r}(D_1 + \cdots + D_r)\bigr)$ is lc.

Proof. Let $\pi : Y' \to Y$ be the blow-up of $v$ and identify its exceptional divisor with $X$. We can write

$$K_{Y'} + (1-\epsilon)X + \pi_*^{-1}\Delta_Y \sim_{\mathbb{Q}} \pi^*\bigl(K_Y + \Delta_Y\bigr)$$

where $\epsilon \geq 0$ since $(Y, \Delta_Y)$ is lc. Restricting to $X$ gives $K_X + \epsilon H + \Delta' \sim_{\mathbb{Q}} 0$ where $\Delta' := \pi_*^{-1}\Delta_Y|_X$ is effective. Thus, by (52), $\epsilon H + \Delta' = cE$. Since $H$ can move, this is only possible if $\epsilon = 0$, that is, if $v$ is an lc center of $(Y, \Delta_Y)$. $\square$

Here are some examples where the assumptions of (52) are satisfied.

Example 54. Let $E_3 \subset \mathbb{P}^2$ be a smooth cubic and $Q \subset E_3$ a set of at least 10 points. Set $X := B_Q\mathbb{P}^2$ and let $E$ be the birational transfrom of $E_1$. Then $K_X + E \sim 0$ and $E^2 < 0$.

Let $C_6 \subset \mathbb{P}^2$ be a rational sextic with 10 nodes and $Q \subset C_6$ the set of nodes. Set $X := B_Q\mathbb{P}^2$ and let $E$ be the birational transform of $C_6$. Then $K_X + \frac{1}{2}E \sim 0$ and $E^2 = -4$.

Example 55 (Non-lc deformations of lc singularities). Let $(X, E)$ be as in (54). Let $p \in X$ be a point, $X_p := B_pX$ the blow up with projection $\pi_p : X_p \to X$ and $E_p \subset X_p$ the birational transform of $E$. If $p \in E$ then $\bigl(X_p, E_p\bigr)$ is still lc and $K_{X_p} + E_p \sim 0$.

On the other hand, if $p$ is not on $E$, we claim that there is no effective $\mathbb{Q}$-divisor $\Delta_p$ on $X_p$ such that $K_{X_p} + \Delta_p \sim_{\mathbb{Q}} 0$. Indeed, this would imply that $K_X + (\pi_p)_*\Delta_p \sim_{\mathbb{Q}} 0$ hence $(\pi_p)_*\Delta_p = E$ by (52). But $p \notin E$, a contradiction.

By taking cones over these surfaces, we get a flat family of 3-fold singularities $C_a\bigl(X_p, H_p\bigr)$ such that

$$\Bigl(C_a\bigl(X_p, H_p\bigr), C_a\bigl(E_p, H_p|_{E_p}\bigr)\Bigr) \quad \text{is lc if } p \in E,$$

but if $p \notin E$, then $\bigl(C_a\bigl(X_p, H_p\bigr), \Delta_p\bigr)$ is not lc, no matter what $\Delta_p$ is.

**Auxiliary results on cones.**

56 (Cones). Let $X \subset \mathbb{P}^n$ be a projective scheme and $f_1, \dots, f_s \in k[x_0, \cdots, x_n]$ generators of its homogeneous ideal.

The *classical affine cone* over $X$ is the variety $C_a(X) \subset \mathbb{A}^{n+1}$ defined by the same equations $(f_1 = \cdots = f_s = 0)$. The closure of the classical affine cone is the *classical projective cone* over $X$, denoted by $C_p(X) \subset \mathbb{P}^{n+1}(x_0{:}\cdots{:}x_n{:}x_{n+1})$. It is defined by the same equations $(f_1 = \cdots = f_s = 0)$ as $X$. (Thus $x_{n+1}$ does not appear in any of the equations.)

Assume that $X$ is normal (resp. $S_2$). Then $C_a(X)$ is normal (resp. $S_2$) iff $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(X, \mathcal{O}(d))$ is onto for every $d \geq 0$ (cf. [**Har77**, Exrc.II.5.14]). Therefore, it is better to define cones for any ample line bundle as follows.

Let $X$ be a projective scheme with an ample line bundle $L$. The *affine cone* over $X$ with conormal bundle $L$ is

$$C_a(X,L) := \operatorname{Spec} \sum_{m\geq 0} H^0(X, L^m),$$

and the *projective cone* over $X$ with conormal bundle $L$ is

$$C_p(X,L) := \operatorname{Proj} \sum_{r\geq 0} \Bigl( \textstyle\sum_{m=0}^{r} H^0(X, L^m)\cdot x_{n+1}^{m-r} \Bigr).$$

Note that if $X \subset \mathbb{P}^N$ and $L = \mathcal{O}_X(1)$ then there is a natural finite morphism $C_p(X, \mathcal{O}_X(1)) \to C_p(X)$ which is an isomorphism away from the vertex. If $X$ is normal then $C_p(X, \mathcal{O}_X(1))$ is the normalization of $C_p(X)$ but $C_p(X, \mathcal{O}_X(1)) \to C_p(X)$ is an isomorphism only if

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \twoheadrightarrow H^0(X, \mathcal{O}_X(m)) \quad \text{is onto } \forall\ m.$$

In particular, the $m = 0$ case shows that $X$ has to be connected.

An advantage of the general notion is that if $X$ is normal (resp. $S_2$) then $C_a(X,L)$ and $C_p(X,L)$ are also normal (resp. $S_2$).

The natural resolutions (blowing up the vertex) are given by

$$BC_a(X,L) := \operatorname{Spec}_X \textstyle\sum_{m\geq 0} L^m$$

in the affine case and by

$$BC_p(X,L) := \operatorname{Proj}_X \sum_{r\geq 0} \Bigl( \textstyle\sum_{m=0}^{r} L^m \otimes \mathcal{O}_X^{m-r} \Bigr) = \operatorname{Proj}_X \textstyle\sum_{r\geq 0} S^r\bigl(L + \mathcal{O}_X\bigr)$$

in the projective case.

Assume that we have a flat family of varieties $X_t \subset \mathbb{P}^n$. Flatness is equivalent to requiring that $\dim R_d(X_t)$ be independent of $t$ for $d \gg 1$; see [**Har77**, III.9.9]. On the other hand, since $\dim R_d\bigl(C_p(X,L)\bigr) = \sum_{i=0}^{d} \dim R_i(X)$, the projective cones $C_p(X_t)$ form a flat family iff

$$\dim R_d\bigl(C_p(X_t)\bigr) = \textstyle\sum_{i=0}^{d} \dim R_i(X_t)$$

is independent of $t$ for $d \gg 1$. Since the $\dim R_i(X_t)$ are upper semi continuous functions of $t$, his implies that $\dim R_i(X_t)$ is independent of $t$ for every $i \geq 0$. This is a source of many interesting examples.

57 (Deformation to the cone over a hyperplane section). Let $X$ be projective, $L$ ample with a nonzero section $s \in H^0(X,L)$ and zero set $H := (s = 0)$. Consider the deformation $\pi : Y \to \mathbb{A}^1_t$ where

$$Y := (s - t x_{n+1} = 0) \subset C_p(X,L) \times \mathbb{A}^1_t.$$

If $t \neq 0$ then we can use $x_{n+1} = t^{-1}s$ to eliminate $x_{n+1}$ and obtain that $Y_t \cong X$. If $t = 0$ then the extra equation becomes $(s = 0)$, thus we get the fiber $Y_0 = C_p(X,L) \cap (s = 0)$. How does this compare with the cone $C_p\bigl(H, L|_H\bigr)$?

The answer is given by the $m = 1$ case of the next result.

PROPOSITION 58. *Let $X$ be a projective scheme with ample line bundle $L$ and $H \subset X$ a Cartier divisor such that $\mathcal{O}_X(H) \cong L^m$ for some $m > 0$. The following are equivalent*

(1) *$C_p\bigl(H, L|_H\bigr)$ is a subscheme of $C_p(X,L)$.*
(2) *$H^1(X, L^d) = 0$ for every $d$.*

Proof. Let $s \in H^0(X, L^m)$ be a section defining $H$ and consider the beginning of the exact sequence

$$\begin{aligned} 0 \to H^0(X, L^{d-m}) &\xrightarrow{s} H^0(X, L^d) \xrightarrow{r_d} H^0(H, L^d|_H) \to \\ &H^1(X, L^{d-m}) \to H^1(X, L^d). \end{aligned} \tag{58.3}$$

The restriction maps $r_d : H^0(X, L^d) \to H^0(H, L^d|_H)$ give a natural morphism $C_p\big(H, L_H\big) \to C_p(X, L)$ which is an embedding iff the $r_d$ are all surjective.

If $H^1(X, L^{d-m}) = 0$ for every $d$ then the $r_d$ are all surjective.

Conversely, assume that $H^1(X, L^{d-m}) \neq 0$ for some $d$. By Serre vanishing, the set of such values $d$ is bounded from above. Thus we can find a $d$ such that $H^1(X, L^{d-m}) \neq 0$ but $H^1(X, L^d) = 0$. Then (58.3) shows that $r_d$ is not surjective hence $C_p\big(H, L|_H\big)$ is not a subscheme of $C_p(X, L)$. □

By iterating this argument, we obtain the following.

Corollary 59. *Let $X$ be projective, CM and $L$ an ample line bundle on $X$. Then $C_a(X, L)$ is CM iff $H^i(X, L^m) = 0 \quad \forall\ m,\ \forall\ 0 < i < \dim X$.* □

Remark 60. A sheaf version in terms of local cohomologies is the following. As in (63) let $F$ be a coherent sheaf on $X$ without 0-dimensional embedded points and $C_a(F, L)$ the corresponding sheaf on $C_a(X, L)$. Let $v \in C_a(X, L)$ be the vertex. Then, for $i \geq 2$,

$$H^i_v\big(C_a(X, L), C_a(F, L)\big) \cong \sum_{m \in \mathbb{Z}} H^{i-1}\big(X, F \otimes L^m\big).$$

For $F = \mathcal{O}_X$ we recover (59).

Proposition 61. *Let $X$ be projective with rational singularities over a field of characteristic zero and $L$ an ample line bundle on $X$. Then $C_a(X, L)$ has rational singularities iff $H^i(X, L^m) = 0 \quad \forall\ m \geq 0,\ \forall\ 0 < i \leq \dim X$.*

(Note that $m$ is arbitrary in (59) while $m \geq 0$ in (61). In characteristic 0, the vanishing for $m < 0$ is guaranteed by Kodaira's theorem.)

Proof. Let $p : BC_a(X, L) \to C_a(X, L)$ be the blow-up of the vertex with exceptional divisor $E \cong X$. Since $BC_a(X, L)$ is a $\mathbb{P}^1$-bundle over $C_a(X, L)$, it has rational singularities. As noted in (48), $C_a(X, L)$ has rational singularities iff $R^i p_* \mathcal{O}_{BC_a(X,L)} = 0$ for $i > 0$.

Let $I \subset \mathcal{O}_{BC_a(X,L)}$ be the ideal sheaf of $E$. By construction,

$$\mathcal{O}_{BC_a(X,L)}/I^m \cong \mathcal{O}_X + L + \cdots + L^{m-1},$$

hence, by the Theorem on Formal Functions,

$$R^i p_* \mathcal{O}_{BC_a(X,L)} = \sum_{m \geq 0} H^i(X, L^m) \quad \text{for } i > 0. \qquad \square$$

Proposition 62. *Assume that $X$ is normal, projective and $L$ is an ample line bundle on $X$. Then*

(1) $\operatorname{Pic}\big(C_a(X, L)\big) = 0$ *and*
(2) $\operatorname{Cl}\big(C_a^*(X, L)\big) = \operatorname{Cl}(X)/\langle L\rangle$ *where $C_a^*(X, L) := C_a(X, L) \setminus$ (vertex) is the punctured affine cone.*

*Let $\Delta_X$ be a $\mathbb{Q}$-divisor on $X$. By pull-back, we get a corresponding $\mathbb{Q}$-divisor $\Delta_{C_a^*(X,L)}$ on $C_a^*(X, L)$ and its closures $\Delta_{C_a(X,L)}$ on $C_a(X, L)$ and $\Delta_{BC_a(X,L)}$ on $BC_a(X, L)$. Assume that $K_X + \Delta_X$ is $\mathbb{Q}$-Cartier. Then*

(3) *$K_{BC_a(X,L)} + \Delta_{BC_a(X,L)} \sim \pi^*(K_X + \Delta_X) - E$ where $\pi : BC_a(X,L) \to X$ is projection from the vertex and $E$ is the exceptional divisor of $p : BC_a(X,L) \to C_a(X,L)$.*

(4) *$m\big(K_{C_a(X,L)} + \Delta_{C_a(X,L)}\big)$ is Cartier iff $\mathcal{O}_X(mK_X + m\Delta_X) \cong L^r$ for some $r$. If this holds then*

$$K_{BC_a(X,L)} + \Delta_{BC_a(X,L)} \sim_{\mathbb{Q}} \big(-\tfrac{r}{m} - 1\big)E.$$

Proof. By construction, $\pi : BC_a(X,L) \to X$ is an $\mathbb{A}^1$-bundle over $E \cong X$, hence $\mathrm{Cl}\big(BC_a(X,L)\big) = \mathrm{Cl}(X)$ and $\mathrm{Pic}\big(BC_a(X,L)\big) = \mathrm{Pic}(X)$. If $M$ is any line bundle on $C_a(X)$ then $p^*M|_E$ is trivial, hence $p^*M$ is trivial and so is $M$, proving (1).

The class group of the punctured cone $C_a^*(X,L) \cong BC_a(X,L) \setminus E$ is computed by the exact sequence

$$\mathbb{Z}\big[\mathcal{O}_{BC_a(X,L)}(E)\big] \to \mathrm{Cl}\big(BC_a(X,L)\big) \to \mathrm{Cl}\big(C_a^*(X,L)\big) \to 0.$$

Since $\mathcal{O}_{BC_a(X,L)}(E)|_E \cong L^{-1}$, we obtain (2).

The projection $\pi : C_a^*(X,L) \to X$ is a $G_m$-bundle. If $t$ is a coordinate on $G_m = \operatorname{Spec} k[t,t^{-1}]$ then the 1-form $dt/t$ is independent of the choice of $t$ since $d(ct)/ct = dt/t$. Thus $\mathcal{O}\big(K_{C_a^*(X,L)/X}\big)$ is the trivial bundle and so

$$K_{C_a^*(X,L)} + \Delta_{C_a^*(X,L)} = \pi^*(K_X + \Delta_X).$$

Thus $K_{BC_a(X,L)} + \Delta_{BC_a(X,L)} \sim \pi^*(K_X + \Delta_X) + mE$ for some $m$ and the adjunction formula gives that $m = -1$.

Combining with the earlier results, we see that $mK_{C_a(X,L)} + m\Delta_{C_a(X,L)}$ is Cartier iff $mK_{C_a^*(X,L)} + m\Delta_{C_a^*(X,L)}$ is trivial iff $\mathcal{O}_X(mK_X + m\Delta_X) \cong L^r$ for some $r$. Then $\pi^*(K_X + \Delta_X) \sim_{\mathbb{Q}} -\frac{r}{m}E$, proving (4). $\square$

63 (Cones of sheaves). Let $F$ be a coherent sheaf on $X$ without 0-dimensional embedded points. Then

$$\sum_{m \in \mathbb{Z}} H^0(X, F \otimes L^m)$$

is a coherent module over $\sum_{m \geq 0} H^0(X, L^m)$. Thus it corresponds to a coherent sheaf $C_a(F,L)$ on the affine cone $C_a(X,L)$.

As before, it is easy to see that $C_a(F,L)$ is CM iff

$$H^i(X, F \otimes L^m) = 0 \quad \forall\, m,\ \forall\, 0 < i < \dim X. \tag{63.1}$$

As an example, consider the case when $S := (xy - zt = 0) \subset \mathbb{P}^3$. The affine cone is $X := C_a(S, \mathcal{O}_S(1)) = (xy - zt = 0) \subset \mathbb{A}^4$, Let $A := (x = z = 0)$ and $B := (x = t = 0)$ be two planes on $X$.

*Claim* 63.2. $\mathcal{O}_X(nA + mB)$ is CM iff either $n = m$ (and then $\mathcal{O}_X(nA + mB)$ is locally free) or $|n - m| = 1$ (in which case $\mathcal{O}_X(nA + mB)$ is not locally free).

Proof. Let $L_A := (x = z = 0)$ and $L_B := (x = t = 0)$ the corresponding lines. Then $\mathcal{O}_X(nA + mB)$ is the cone over $\mathcal{O}_S(nL_A + mL_B)$ and $\mathcal{O}_S(1) \sim L_A + L_B$. By (63.1) we need to check when

$$H^1(S, \mathcal{O}_S((n-r)L_A + (m-r)L_B)) = 0 \quad \forall r.$$

By the Küneth formula,

$$\begin{aligned} H^1(S, \mathcal{O}_S((n-r)L_A + (m-r)L_B)) \;&=\; H^1(\mathbb{P}^1, \mathcal{O}(n-r)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(m-r)) + \\ &\phantom{=}\; H^0(\mathbb{P}^1, \mathcal{O}(n-r)) \otimes H^1(\mathbb{P}^1, \mathcal{O}(m-r)). \end{aligned}$$

If $|n-m| \geq 2$ then we get a nonzero term for $r = \max\{n, m\}$. Otherwise we always get zero.

## 3. Surface Singularities of the Minimal Model Program

In this section we study the surface singularities that appear in the minimal model program. While our main interest is over fields of characteristic zero, the argument we present works in arbitrary characteristic and even for excellent surfaces. The few foundational questions needed for this generality are discussed at the end of the section.

The results give a rather complete description of log canonical surface singularities in terms of the combinatorial structure of the exceptional curves on the minimal resolution (64). In the theory of minimal models this seems to be the most useful. Also, it turns out, the answer is independent of the characteristic.

One can then go further and determine the completed local rings of the singularities. Most of these have been worked out in characteristic 0 but subtle differences appear in positive characteristic [**Lip69**]. We will not discuss this aspect here.

THEOREM 64 (Log minimal resolution of surface pairs). *Let $Y$ be a 2-dimensional, normal, excellent scheme, $B_i \subset X$ distinct, irreducible Weil divisors and $\Delta := \sum b_i B_i$ a linear combination of them with $0 \leq b_i \leq 1$ for every $i$.*

*Then there is a proper birational morphism $f : X \to Y$ such that*

(1) *$X$ is regular,*
(2) *$K_X + f_*^{-1}\Delta$ is $f$-nef,*
(3) *$\operatorname{mult}_x f_*^{-1}\Delta \leq 1$ for every $x \in X$,*
(4) *$f_*^{-1}\lfloor\Delta\rfloor = \sum_{i:b_i=1} f_*^{-1}B_i$ is regular, and*
(5) *we can choose either of the following conditions to hold:*
   (a) *the support of $\operatorname{Ex}(f) + f_*^{-1}\lfloor\Delta\rfloor$ has a node (101) at every point of $\operatorname{Ex}(f) \cap f_*^{-1}\lfloor\Delta\rfloor$, or*
   (b) *$K_X + (1-\epsilon)f_*^{-1}\Delta$ is $f$-nef for $0 \leq \epsilon \ll 1$.*

Proof. Resolution is known for excellent surfaces (cf. [**Sha66**]), hence there is a proper birational morphism $f_1 : X_1 \to Y$ such that $X_1$ is regular, the support of $f_*^{-1}\lfloor\Delta\rfloor$ is regular and the support of $\operatorname{Ex}(f) + f_*^{-1}\lfloor\Delta\rfloor$ has only nodes.

If $K_{X_1} + (f_1)_*^{-1}\Delta$ is not $f_1$-nef then there is an irreducible and reduced exceptional curve $C \subset X_1$ such that $C \cdot (K_{X_1} + (f_1)_*^{-1}\Delta) < 0$. Since $C \cdot (f_1)_*^{-1}\Delta \geq 0$, this implies that $C \cdot K_{X_1} < 0$. By the Hodge Index Theorem (95), $(C \cdot C) < 0$, thus $\omega_C$ is anti-ample.

If we are in characteristic 0, then $C$ is geometrically reduced and every geometric irreducible component of $C$ is a smooth rational with self intersection $-1$. Thus we can contract $C$ by Castelnuovo's theorem. Furthermore, $(C \cdot (f_1)_*^{-1}\Delta) < 1$ which shows that we do not contract any curve which meets $f_*^{-1}\lfloor\Delta\rfloor$ and the multiplicity of the birational transform of $\Delta$ is still $\leq 1$ at every point. Thus, after contracting $C$, we get $f_2 : X_2 \to Y$ and the conditions (3–4) still hold. We can continue until we get $f : X \to Y$ as required.

There is only one case that forces us to choose between the two alternatives in (5). If

$$C \cdot (K_{X_1} + (f_1)_*^{-1}\Delta) = 0 \quad \text{but} \quad C \cdot (K_{X_1} + (1-\epsilon)(f_1)_*^{-1}\Delta) < 0$$

then we can still contract $C$ but the image of $\lfloor \Delta \rfloor$ may pass through a singular point of $\operatorname{Ex}(f)$. So we have to decide which alternative we want. Both have some advantages.

We check in (99) that Castelnuovo's contraction theorem holds even if $C$ is not geometrically reduced. The rest of the argument then goes as before. $\square$

65 (Dual graphs). Let $X$ be a regular surface and $C = \cup C_i$ a proper curve on $X$. It is frequetly very convenient to represent the curve $C$ by a graph $\Gamma$ whose vertices are the irreducible components of $C$ and two vertices are connected by an edge iff the corresponding curves intersect. For exceptional curves, the self-intersection numbers $(C_i \cdot C_i)$ are negative, and usually we use the number $-(C_i \cdot C_i)$ to represent a vertex and add the arithmetic genus of $C_i$ as extra marking. We write the intersection number $(C_i \cdot C_j)$ on an edge if this number is different from 1. This graph, with various extra information added, is called the *dual graph* of the reducible curve $C = \cup C_i$. A dual graph is called *negative definite* if the intersection form $(C_i \cdot C_j)$ is negative definite.

For log canonical singularities, most of the exceptional curves $C_i$ are smooth and rational and the dual graphs have few edges, so the picture is rather transparent.

Let $\det(\Gamma)$ denote the determinant of the intersection matrix of the dual graph. This matrix is the negative of the intersection form, hence positive definite for exceptional curves. For instance, if $\Gamma = \{2 \ - \ 2 \ - \ 2\}$ then

$$\det(\Gamma) = \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 4.$$

Let $B$ be another divisor on $X$ which does not contain any of the $C_i$. The *extended dual graph* $(\Gamma, B)$ has an additional vertex connected to $C_i$ if $(B \cdot C_i) \neq 0$. The new verties are repesented by the intersection number $(B \cdot C_i)$.

One can also think about (extended) dual graphs as combinatorial objects, without reference to any particular collection of curves. (Although one can see that every (extended) dual graph arises from a collection of curves on a smooth projective surface over $\mathbb{C}$.)

66. Let $f : X \to Y$ be a log minimal resolution constructed in (64), $y \in Y$ a point and $\Gamma = \Gamma(y \in Y)$ the dual graph of the curves $\operatorname{red} f^{-1}(y) = \cup_i C_i$. Since the intersection matrix $(C_i \cdot C_j)$ is invertible, there is a unique $\Delta = \Delta(y, Y, B) := \sum d_j C_j$ such that

$$\big(\Delta \cdot C_i\big) = -\big((K_X + f_*^{-1}B) \cdot C_i\big) \qquad \forall i. \tag{66.1}$$

Using the adjunction formula, we can rewrite these equations as

$$\big(\Delta \cdot C_i\big) = (C_i \cdot C_i) + 2 - 2p_a(C_i) - \big(f_*^{-1}B \cdot C_i\big) \qquad \forall i. \tag{66.2}$$

Note that these equations make sense for any extended dual graph $(\Gamma, B)$, even if the graph does not arise from a collection of curves on a surface.

By (64.2) the right hand side of (66.1) is always $\leq 0$ for a log-minimal resolution. Therefore, we call an extended dual graph $(\Gamma, B)$ *log-minimal* if the right hand sides of (66.2) are all $\leq 0$.

A simple linear algebra lemma (97.5) gives the following.

*Claim 66.3.* Let $(\Gamma, B)$ be a connected, negative definite, log-minimal extended dual graph. Then the system (66.2) has a unique solution $\Delta = \sum d_j C_j$. Furthermore

(1) either $d_j = 0$ for every $j$,
(2) or $d_j > 0$ for every $j$. □

If $K_Y + B$ is $\mathbb{Q}$-Cartier (which will always be the case for us) then $d_j$ is the negative of the discrepancy $a(C_j, Y, B)$ defined in (6). Therefore, an extended dual graph $(\Gamma, B)$ is called *numerically canonical* (resp. *numerically log terminal* or *numerically log canonical*) if it is connected, negative definite, log-minimal and $d_j \leq 0$ for every $j$ (resp. $d_j < 1$ for every $j$ or $d_j \leq 1$ for every $j$).

67 (Numerically log canonical). Let $Y$ be a normal, excellent surface and $B$ an effective $\mathbb{Q}$-divisor. We say that $(Y, B)$ is *numerically log canonical* at a point $y \in Y$ if there is a log minimal resolution $f : X \to Y$ as in (64) such that the extended dual graph of $f^{-1}(y) = \cup_i C_i$ is numerically log canonical.

The notion of a numerically log canonical pair is a temporary convenience. It is more general than log canonical in 3 aspects.

We assume the discrepancy condition $a(E_i, Y, B) \geq -1$ only for those curves that appear on a given resolution. Even over $\mathbb{C}$, this leads to a few extra cases which are, however, easy to enumerate.

By working directly on $X$, we do not worry at the beginning whether $K_Y + B$ is $\mathbb{Q}$-Cartier or not. This turns out to be automatic in the surface case.

In positive and in mixed characteristic, we work with regular schemes instead of smooth schemes. The whole minimal model program is very geometric in nature, so it may be quite unnatural to consider schemes which are regular but not smooth. In the relative setting, however, such fibers inevitably appear. My sole aim here is to note that some of the basic results hold for regular surfaces. These results are needed for the study of semi stable morphisms in positive and mixed characteristics.

It may well turn out, however, that in higher dimensions regular schemes are too pathological for a meaningful minimal model program.

68 (Plan for the classification). Instead of giving a direct classification of germs of log canonical pairs $(y \in Y, B)$, we aim to develop a description of numerically log canonical extended dual graphs.

This approach is relatively quick and the answer is independent of the characteristic. A disadvantage is that it is not always easy to go from the dual graph to the actual surface germ $(y \in Y)$. Fortunately, it turns out that for our applications the extended dual graph contains all the necessary information.

First we classify the terminal and canonical cases geometrically. Then we describe the cases when the boundary $B$ contains some curves with coefficient 1. This is the most important case for our moduli problems.

Finally we study the dual graps combinatorially, essentially by trying to solve the system of equations (66.2). A full solution seems hopeless, but we are able to get a good understanding of how the solutions change if we change the dual graph. Ultimately, we end up with a short list.

69 (Classification using index 1 covers). The traditional classification of log canonical surface singularities over $\mathbb{C}$ proceeds as follows.

If $K_S$ is a Cartier divisor, then every curve appears in the discrepancy divisor $\Delta$ with integral coefficient. Thus, by (66.3), either $\Delta = 0$ (and hence $\Gamma$ is Du Val), or $\Delta = \sum C_i$, the sum of all curves in $\Gamma$

Thus the system of equations (66.2) becomes

$$\#(\text{neighbors of } C_i) = 2 - 2p_a(C_i) \quad \forall i.$$

We have 2 possibilities. Either $\Gamma$ consists of a single elliptic or nodal rational curve (as in (77)) or $\Gamma$ consists smooth rational curves, each with 2 neighbors (as in (77.3)).

In characteristic zero, the covering method of Reid shows that every lc surface singularity is a quotient of a lc surface singularity whose canonical class is Cartier, see [**Rei80**]. Thus one can classify all lc surface singularities by describing all group actions on the singularities listed in (71) and (77).

In positive characteristic this method has problems when the characteristic divides the order of the group. For partial results, see [**Kaw94**] with corrections in [**Kaw99**].

Note that quotients of $\mathbb{A}^2$ by $\mathbb{Z}/p$ can be very pathological in characteristic $p$; see [**Art75**].

**Terminal and canonical pairs.**

Let $(s \in S, B)$ be the germ of a normal surface pair that is numerically canonical. By definition, this means that $\Delta := \Delta(s \in S, B)$ defined by the equations (66.2) contains every curve with $\leq 0$ coefficient. On the other hand, by (66.3), $\Delta$ conatains every curve with $\geq 0$ coefficient. Thus $\Delta = 0$.

THEOREM 70. *$(s \in S, B)$ is numerically canonical iff*

(1) *either $s \in S$ is regular and $\operatorname{mult}_s B \leq 1$,*
(2) *or $s \notin \operatorname{Supp} B$ and there is a resolution $f : T \to S$ such that $\mathcal{O}_T(K_{T/S}) \cong f^*\mathcal{O}_S$ (cf. (71)).*

*Proof.* Let $(s \in S, B)$ be numerically canonical with minimal resolution $f : T \to S$ satisfying the alternative (64.5.b). Thus $K_T + f_*^{-1}B$ is numerically $f$-trivial and $K_T + (1-\epsilon)f_*^{-1}B$ is numerically $f$-nef. Thus $-f_*^{-1}B$ is numerically $f$-nef. This can happen only in two degenerate ways: either $f$ is an isomorphism or $B = 0$. In the latter case, the isomorphism $\mathcal{O}_T(K_{T/S}) \cong f^*\mathcal{O}_S$ follows from (75).

Conversely, assume that $S \in S$ is regular and $\operatorname{mult}_s B \leq 1$. Then $a(E_1, S, B) = 1 - \operatorname{mult}_s B \geq 0$ as above. If $E$ is any other execeptional curve over $S$ then by (12) and (10)

$$a(E, S, B) = a(E, S_1, \pi_*^{-1}B + (\operatorname{mult}_s B - 1)E_1) \geq a(E, S_1, \pi_*^{-1}B).$$

Since $\operatorname{mult}_p \pi_*^{-1}B \leq \operatorname{mult}_{\pi(p)} B \leq 1$, we are done by induction on the number of blow ups necessary to reach $E$. $\square$

EXAMPLE 71 (Du Val singularities). Consider case (70.2) over an algebraically closed field. If the exceptional curves are $E_i$, then their self intersections are computed from $(E_i \cdot E_i) = (K_T + E_i) \cdot E_i = 2p_a(E_i) - 2$. By (95), $(E_i \cdot E_i) < 0$, hence $p_a(E_i) = 0$, $E_i \cong \mathbb{P}^1$ and $(E_i \cdot E_i) = -2$. Since the intersection form is negative definite, this implies that $(E_i \cdot E_j) \in \{0, 1\}$ for $i \neq j$.

The classification of such dual graphs is easy to do by hand and it has been done many times from different points of view. The first time probably as part of the classification of root systems of simple Lie algebras; giving rise to their names.

The corresponding singularities are called *Du Val* singularities or *rational double points* or *simple* surface singularities. See [**KM98**, Sec.4.2] or [**Dur79**] for more information. (The equations below are correct in characteristic zero. The dual graphs are correct in every characteristic.)

$A_n$: $x^2+y^2+z^{n+1}=0$, with $n \geq 1$ curves in the dual graph:

$$2 \;-\; 2 \;-\; \cdots \;-\; 2 \;-\; 2$$

$D_n$: $x^2+y^2z+z^{n-1}=0$, with $n \geq 4$ curves in the dual graph:

$$\begin{array}{ccccccccc} & & 2 & & & & & & \\ & & | & & & & & & \\ 2 & - & 2 & - & \cdots & - & 2 & - & 2 \end{array}$$

$E_6$: $x^2+y^3+z^4=0$, with 6 curves in the dual graph:

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & & | & & & & \\ 2 & - & 2 & - & 2 & - & 2 & - & 2 \end{array}$$

$E_7$: $x^2+y^3+yz^3=0$, with 7 curves in the dual graph:

$$\begin{array}{ccccccccccc} & & & & 2 & & & & & & \\ & & & & | & & & & & & \\ 2 & - & 2 & - & 2 & - & 2 & - & 2 & - & 2 \end{array}$$

$E_8$: $x^2+y^3+z^5=0$, with 8 curves in the dual graph:

$$\begin{array}{ccccccccccccc} & & & & 2 & & & & & & & & \\ & & & & | & & & & & & & & \\ 2 & - & 2 & - & 2 & - & 2 & - & 2 & - & 2 & - & 2 \end{array}$$

**The reduced boundary.**

For a log canonical pair $(S, B=\sum b_iB_i)$, the singularities of $\operatorname{Supp} B$ can be arbitrary if the coefficients $b_i$ are small. By contrast, if the $b_i$ are bounded from below, we end up with a restricted class of singularities. In the exreme case, when all the $b_i$ equal 1, we show that $B$ has only ordinary nodes. The following theorem is a more precise version of this assertion.

THEOREM 72. *Let $(S, B+B')$ be numerically lc and $B=\sum B_i$ a sum of curves, all with coefficients 1. Then, for any $s \in S$,*

(1) *either $B$ is regular at $s$,*
(2) *or $B$ has a node (101) at $s$,* $\operatorname{Supp} B'$ *does not contain $s$ and there is at least 1 exceptional divisor $E$ with discrepancy 1 whose center is $s$.*

Proof. The question is local on $S$. Fix a point $s \in S$ and let $f: T \to S$ be a log minimal resolution as in (64.5.a).

Write $K_T+B_T+\Delta_T+f_*^{-1}B' \sim_{\mathbb{Q}} f^*(K_S+B+B')$ where $B_T := f_*^{-1}B$ is regular by (64.4). Note that $\Delta_T=\sum d_iC_i$ is effective by (66.3), $f$-exceptional and thus $0 \leq d_i \leq 1$ for every $i$. We can rewrite the linear equivalence as

$$-B_T \sim_{\mathbb{Q}} K_T+\Delta_T+f_*^{-1}B'-f^*(K_S+B+B'). \tag{72.3}$$

Pushing forward the exact sequence

$$0 \to \mathcal{O}_T(-B_T) \to \mathcal{O}_T \to \mathcal{O}_{B_T} \to 0,$$

we get the exact sequence

$$\mathcal{O}_S \cong f_*\mathcal{O}_T \to f_*\mathcal{O}_{B_T} \to R^1 f_*\mathcal{O}_T(-B_T).$$

By the general Grauert-Riemenschneider vanishing theorem (98), either $R^1 f_*\mathcal{O}_T(-B_T) = 0$ or $\Delta_T = C_1 + \cdots + C_n$ where the $C_i$ are all the exceptional curves and $s \notin \operatorname{Supp} B'$. We consider these two possibilities separately.

*Case 1.* If $R^1 f_*\mathcal{O}_T(-B_T) = 0$ then the composite

$$\mathcal{O}_S \to \mathcal{O}_B \hookrightarrow f_*\mathcal{O}_{B_T} \quad \text{is surjective.}$$

Thus $B \cong B_T$ and so $B$ is regular.

*Case 2.* Otherwise we prove that $B$ has a node at $s$. We can drop $B'$ and rewrite (72.3) as

$$-B_T - \Delta_T \sim_{\mathbb{Q}} K_T - f^*(K_S + B). \tag{72.4}$$

Then $R^1 f_*\mathcal{O}_T(-B_T - \Delta_T) = 0$ and, as above, we get that $I_{s,S} = f_*\mathcal{O}_T(-\Delta_T) \to f_*\mathcal{O}_{B_T}(-\Delta_T|_{B_T})$ is surjective. By (64.4), $\mathcal{O}_{B_T}(-\Delta_T|_{B_T})$ is the ideal of the closed points of $B_T$. In particular,

$$\mathcal{O}_B(-s) = f_*\mathcal{O}_{B_T}\big(-\operatorname{red} f^{-1}(s)\big). \tag{72.5}$$

Since $\big(C_i \cdot (K_T + B_T + \sum C_i)\big) = 0$ for every $i$, the adjunction formula gives that

$$\deg \omega_{C_i} = -\textstyle\sum_{j\neq i}(C_i \cdot C_j) - (C_i \cdot B_T) \quad \forall\, i. \tag{72.6}$$

Assume first that we are over an algebraically closed field. Then each $C_i \cong \mathbb{P}^1$ and the dual graph is

$$B_1 - C_1 - \cdots - C_n - B_2$$

Thus $B$ has 2 irreducible components and, by (72.5), the 2 branches intersect transversally.

Essentially the same argument works in general, but one has to be more careful, especially over imperfect fields.

We have already dealt with the second condition of (101.1).

Let $k(s)$ be the residue field of $s \in S$. In order to apply (101.1) we need to show in addition that

$$(B_T \cdot \Delta_T) = \dim_{k(s)}\Big(\mathcal{O}_{B_T}/\mathcal{O}_{B_T}(-\Delta_T|_{B_T})\Big) \leq 2.$$

By (100), for each $i$, the right hand side of (72.6) has either 1 nonzero term or 2 nonzero terms which are equal. This gives two possibilities for the dual graph:

$$\begin{array}{l} B_1 \overset{r}{-} C_1 \overset{r}{-} \cdots \overset{r}{-} C_n \overset{r}{-} B_2 \quad \text{or} \\ B_1 \overset{r}{-} C_1 \overset{r}{-} \cdots \overset{r}{-} C_n \end{array}$$

where $r$ denotes the intersection numbers. The curves $C_i$ for $i \leq n$ in the first case (resp. $i \leq n-1$ in the second case) are as in (100.3.b) and in the second case the last curve $C_n$ can be of type (100.3.a). Thus

$$(B_1 \cdot C_1) = (C_1 \cdot C_2) = \cdots = (C_{n-1} \cdot C_n) = \begin{cases} \dim_{k(s)} H^0(C_n, \mathcal{O}_{C_n}) & \text{in case (100.3.b),} \\ 2\dim_{k(s)} H^0(C_n, \mathcal{O}_{C_n}) & \text{in case (100.3.a).} \end{cases}$$

Working through the exact sequences

$$0 \to \mathcal{O}_{C_i}(-C_i \cap C_{i+1}) \to \mathcal{O}_{C_i + \cdots + C_n} \to \mathcal{O}_{C_{i+1} + \cdots + C_n} \to 0$$

we conclude that $H^0(\mathcal{O}_{C_1+\cdots+C_n}) = H^0(\mathcal{O}_{C_n})$. From (72.4) and (98) we infer that $R^1 f_*\mathcal{O}_T(-\Delta_T) = 0$, hence $k(s) \cong H^0(\mathcal{O}_{\Delta_T}) \cong H^0(\mathcal{O}_{C_n})$.

Thus, in both cases, $(B_T \cdot \Delta_T) \le 2$. $\square$

REMARK 73. We have also proved that in case (72.2), the extended dual graph is

$$\bullet \ - \ c_1 \ - \ \cdots \ - \ c_n \ - \ \bullet \tag{73.1}$$

where $\bullet$ denotes a component of $B$. As a degenerate case we allow $\bullet \ - \ \bullet$ which corresponds to the simple normal crossing point $\big((xy = 0), \mathbb{A}^2\big)$.

Conversely, for any $c_1, \dots, c_n \ge 2$ the singularities with this dual graphs are lc. The discrepancies are $-1$ for the $n$ vertices marked $c_i$.

We will see later that in case (72.2) the extended dual graph is either

$$\bullet \ - \ c_1 \ - \ \cdots \ - \ c_n \tag{73.2}$$

or

$$\begin{array}{cccccccc} & & & & & & & 2 \\ & & & & & & / & \\ \bullet & - & c_1 & - & \cdots & - & c_n & \\ & & & & & & \backslash & \\ & & & & & & & 2 \end{array} \tag{73.3}$$

In the latter case, the discrepancies are $-1$ for the $n$ vertices marked $c_i$ and $-1/2$ for the two vertices marked 2.

As a corollary, using the description of the deformations of nodes given in (101.2) we obtain the following, which gives a complete description of the codimension 1 behavior of fibers in semi stable families (**??**?).

COROLLARY 74. *Notation and assumptions as in (72). Assume in addition that $B$ is a Cartier divisor. Then either $B$ and $S$ are both regular at $s$, or $B$ has a node at $s$ and $S$ has a double point of embedding dimension 3 at $s$.*

*Furthermore, if $S$ is over an algebraically closed field $k$ of characteristic 0, then the completion of $\mathcal{O}_{s,S}$ is isomorphic to $k[[x,y,z]]/(xy - z^n)$ and $\hat{B} = (z = 0)$.* $\square$

**First examples of log canonical singularities.**

PROPOSITION 75. *Let $(y \in Y, B)$ be a numerically log canonical surface germ, $f : X \to Y$ a log minimal resolution with exceptional curves $C_i$. Let $\Delta := \Delta(y \in Y, B)$ be the discrepancy divisor as in (66). Then*

1. *either $y \notin \operatorname{Supp} B$ and $\Delta = \sum_i C_i$ (cf. (77)),*
2. *or $(y \in Y)$ is a rational singularity, that is, $R^1 f_*\mathcal{O}_X = 0$.*

Proof. Note that $K_X + f_*^{-1}B + \Delta$ has zero intersection number with every exceptional curve. Thus the Grauert-Riemenschneider vanishing (98) applies to $L = \mathcal{O}_X$ and gives (2), unless $f_*^{-1}B \cdot C_i = 0$ for every $i$ (and hence $y \notin \operatorname{Supp} B$) and every exceptional curve appears in $\Delta$ with coefficient 1. $\square$

PROPOSITION 76. *Let $(y \in Y)$ be a rational surface singularity and $f : X \to Y$ a log minimal resolution with exceptional curves $C_i$.*

1. *Let $L$ be a line bundle such that $\deg_{C_i} L = 0$ for every $i$. Then $f_*L$ is locally free and $L \cong f^*(f_*L)$.*
2. *$Y$ is $\mathbb{Q}$-factorial at $y$. More precisely,* $\det(\Gamma)D$ *is Cartier for any Weil divisor $D$ on $S$, where $\Gamma$ is the dual graph of $f^{-1}(y)$.*

Proof. If $Z$ is any 1-cycle with support in $f^{-1}(y)$, then

$$R^1 f_*\mathcal{O}_X \to H^1(Z, \mathcal{O}_Z) \to R^2 f_*\mathcal{O}_X(-Z) = 0$$

shows that $H^1(Z, \mathcal{O}_Z) = 0$. Thus, by [**Mum66**, Lect.24], $\operatorname{Pic}(Z) \cong \operatorname{Pic}(\operatorname{red} Z)$ and a line bundle $M$ on $Z$ is determined by the degrees $\deg_{C_i} M$ for $C_i \subset Z$. Hence if $(L \cdot C_i) = 0$ for every $i$ then $L|_Z \cong \mathcal{O}_Z$ for every $Z$. By the formal function theorem, $f_*L$ is locally free at $y$. Thus the natural map $f^*(f_*L) \to L$ is an isomorphism.

Let $D$ be any Weil divisor on $S$ and $f_*^{-1}D$ its birational transform on $X$. By the Hodge index theorem (95) and by Kramer's rule, the system of equations

$$\textstyle\sum_i a_i\bigl(C_i \cdot C_j\bigr) = \bigl(f_*^{-1}D \cdot C_j\bigr) \quad \forall\ j$$

have a unique solution and $\det(\Gamma) \cdot a_i \in \mathbb{Z}$. Apply the fist part to $L = \mathcal{O}_X\bigl(\det(\Gamma) \cdot f_*^{-1}D + \sum(\det(\Gamma) \cdot a_i)C_i\bigr)$ to get that $\mathcal{O}_X\bigl(\det(\Gamma) \cdot D\bigr)$ is locally free, hence $\det(\Gamma) \cdot D$ is Cartier. $\square$

Next we discuss the main examples of surface singularities that are lc but not lt.

Example 77 (Numerically elliptic/cusp singularities). By definition, these are the singularities $(p \in S, B)$ where $B = 0$ and every exceptional curve appears in $\Delta := \Delta(y \in Y, B)$ with coefficient 1.

One should treat elliptic/cusp singularities as one class, but traditionally they have been viewed as two distinct types. For these singularities one can write the equations (65.1) as

$$\deg \omega_{C_i} = -\textstyle\sum_{i \neq j} (C_i \cdot C_j).$$

The usual approach distinguishes 2 solutions:

(1) $\Gamma$ consists of a single curve $C$ with $\deg \omega_C = 0$. These are called numerically (simple) *elliptic*. If we are over an algebraically closed field, then there are 3 subcases:
   (a) a smooth elliptic curve (called a *simple elliptic singularity*),
   (b) a nodal rational curve, or
   (c) a cuspidal rational curve (two more blow ups show that this is not lc).
(2) $\Gamma$ consists of at least 2 curves $C_i$ with $\deg \omega_{C_i} < 0$ for every $i$. These are called numerically *cusp*[1] singularities. (Usually case (2.b) is also considered a cusp singularity.)

The distinction between (simple) elliptic and cusp singularities is made mostly for historic reasons, and over a nonclosed field $k$ the two concepts are mixed together. Indeed, we can have a cusp over $\bar{k}$ where the Galois group of $\bar{k}/k$ acts transitively on the exceptional curves. In that case, we have a numerically elliptic singularity over $k$ but a cusp over $\bar{k}$.

For a cusp over an algebraically closed field, every curve $C_i$ is smooth, rational and intersects the rest with multiplicity 2. Thus the dual graph is a circle of smooth

[1]Cusps of curves and cusps of surfaces are quite unrelated. Usually there is no danger of confusion.

rational curves:

$$\begin{array}{ccccccc} & & c_n & - \cdots - & c_{r+1} & & \\ & / & & & & \backslash & \\ c_1 & & & & & & c_r \\ & \backslash & & & & / & \\ & & c_2 & - \cdots - & c_{r-1} & & \end{array} \tag{77.3}$$

where $n \geq 3$. The $n = 1, 2$ cases are somewhat special. For $n = 2$ we have 2 curves which intersect at 2 distinct points. For $n = 1$ we get a nodal rational curve; this was already considered above.

The intersection form is negative definite if at least one of the $c_i \geq 3$, as shown by the vector $\sum C_i$ (97.4.2).

There are also 2 degenerate cases. 2 curves can be tangent at a point or 3 curves can meet in 1 point. These are numerically log canonical but not log canonical.

EXAMPLE 78 ($\mathbb{Z}/2$-quotient of a cusp or simple elliptic). For any $c_1, \dots, c_n \geq 2$, with at least one $c_i \geq 3$, the singularities with dual graphs below are lc, where $\bullet$ denotes a component of $B$ with coefficient 1.

The discrepancies are $-1$ for the $n$ vertices marked $c_i$ and $-\frac{1}{2}$ for the vertices marked 2.

$$\begin{array}{ccccccc} 2 & & & & & & 2 \\ & \backslash & & & & / & \\ & & c_1 & - \cdots - & c_n & & \\ & / & & & & \backslash & \\ 2 & & & & & & 2 \end{array} \tag{78.1}$$

(The name comes from the fact that, when the characteristic is different from 2, the corresponding singularity is a $\mathbb{Z}/2$-quotient of a cusp (for $n \geq 2$) or a $\mathbb{Z}/2$-quotient of a simple elliptic singularity (for $n = 1$).)

**Classification of log canonical singularties, I..**

In what follows, we classify lc surface germs $(s \in S, B)$ by focusing on the dual graph, following the method of Alexeev [**K$^+$92**, Sec.2]. Besides working in any characteristic, this approach is also better at dealing with the boundary $B$.

79 (Classification method). Let $(s \in S, B)$ be a surface singularity, $f : T \to S$ a log minimal resolution as in (64) with exceptional curves $C_i$ and $\Gamma = \Gamma(s \in S, B)$ the extended dual graph (65) of the curves $\sum C_i$ and $f_*^{-1}B$. More generally, we consider all connected, numerically log canonical extended dual graphs $(\Gamma, B)$.

In what follows, we consider the equations (66.2) and try to find conditions on the right hand side which force $d_j > 1$ for some $j$. This would then show that $(\Gamma, B)$ is not lc.

We will identify various small subsets of the equations which lead to a contradiction, no matter what the remaining equations are. After 3 such steps we are left with only a handful of cases. These will then be studied more carefully.

We use the following form of (97.4):

*Claim* 79.1 If $(\Delta \cdot C_i) \leq (\Delta' \cdot C_i)$ for every $i$ then $\Delta \geq \Delta'$. Furthermore, if one of the inequalities is strict then $\Delta \gg \Delta'$. (That is, $\Delta - \Delta'$ contains every curve with poisitive coefficient.) In particular, if $(\Delta \cdot C_i) \leq 0$ for every $i$ then $-\Delta$ is effective. $\square$

As a first application, we see how $\Delta(\Gamma, B)$ changes if we change $B$.

79.2 (Changing $B$.) From (79.1) we see that if $B' \lneq B$ then $\Delta(\Gamma, B') \ll \Delta(\Gamma, B)$.

The most important comparison is the following.

79.4 (Passing to a subgraph.) Let $(\Gamma', B')$ be graph obtained from $(\Gamma, B)$ by deleting some vertices and edges. Let $J$ (resp. $J'$) be the set of vertices in $\Gamma$ (resp. $\Gamma'$). If $\Delta(\Gamma, B) = \sum_{j \geq J} d_j C_j$ then, for every $i \in J'$,

$$\big(\sum_{j \in \Gamma'} d_j C_j \cdot C_i\big) = (C_i \cdot C_i) - \deg \omega_{C_i} - (B' \cdot C_i) - \big(\sum_{j \notin \Gamma'} d_j C_j \cdot C_i\big) - \big((B - B') \cdot C_i\big).$$

Thus we see that $\Delta(\Gamma', B') \leq \sum_{j \in \Gamma'} d_j C_j$ and in fact

$$\Delta(\Gamma', B') \ll \sum_{j \in \Gamma'} d_j C_j,$$

unless $d_j = 0$ for every $j \in J \setminus J'$. That is, when $(\Gamma, B)$ is canonical.

In particular, if $(\Gamma', B')$ is not log terminal then $(\Gamma, B)$ is not log canonical. As a first consequence we obtain the following (which also follows from (75)).

*Corollary* 79.4 Let $(\Gamma, B)$ be a log canonical extended dual graph not yet enumerated in (77). Then every $C_i$ is a smooth rational curve and $\Gamma$ is a tree. $\square$

79.5 (Increasing $c_i$.) Let $\Gamma'$ be the graph obtained from $\Gamma$ by changing $c_1 = -(C_1 \cdot C_1)$ to $c_1' > c_1$. All other intersections are unchanged. Let $C_i'$ denote the curves in $\Gamma'$.

If $\Delta(\Gamma, B) = \sum d_j C_j$ then

$$\big(\sum d_j C_j' \cdot C_i'\big) = (C_i' \cdot C_i') - \deg \omega_{C_i} - (B' \cdot C_i') - \begin{cases} (1 - d_1)(c_1' - c_1) & \text{if } i = 1, \text{ and} \\ 0 & \text{if } i > 1. \end{cases}$$

Thus if $d_1 < 1$ and $c_1' > c_1$ then $\Delta(\Gamma, B) \ll \Delta(\Gamma', B')$. In particular, we see that if in the dual graph (78) we increase any of the vertices marked 2, then we get a dual graph that is not log canonical. By (79.4), any dual graph containing such a subgraph is also not log canonical.

We can summarize our results as follows:

Corollary 80. *Let $\Gamma$ be a log canonical extended dual graph not yet enumerated in (77), (78) or (73). Then every $C_i$ is a smooth rational curve and one of the following holds.*

(1) *(Cyclic quotient) $\lfloor B \rfloor$ is smooth and $(\Gamma, \lfloor B \rfloor)$ is*

$$\bullet \ - \ c_1 \ - \ \cdots \ - \ c_n \quad \text{or} \quad c_1 \ - \ \cdots \ - \ c_n$$

(2) *(Other quotient) $\lfloor B \rfloor = 0$ and $\Gamma$ is a tree with 3 branches:*

$$\begin{array}{ccccccccc} \circ & - & \cdots & - & \circ & - & \cdots & - & \circ \\ &&&& | &&&& \\ &&&& \vdots &&&& \\ &&&& | &&&& \\ &&&& \circ &&&& \end{array}$$

We deal with the resulting 2 classes separately.

**Cyclic quotients and the different.**

DEFINITION 81 (Cyclic quotients). Let $k$ be a field and fix natural numbers $1 \leq a, b \leq r$. Consider a $\mathbb{Z}/r$-grading of $k[x,y]$ given by $w(x^n y^m) = an + bm \mod r$. Let $R_{a,b} := \{f \in k[x,y] : w(f) = 0\}$ denote the ring of degree 0 elements. We use the notation

$$\mathbb{A}^2/\tfrac{1}{r}(a,b) := \operatorname{Spec}_k R_{a,b}.$$

To explain this notation, assume first that $\operatorname{char} k \not| r$ and let $\mu_r = \langle \epsilon \rangle$ be the group of $r$th roots of unity. The above grading is equivalent to a $\mu_r$-action given by $(x,y) \mapsto (\epsilon^a x, \epsilon^b y)$. Then $R_{a,b}$ is the ring of invariants of this action. (If $\operatorname{char} k | r$ then $\mu_r$ is a nonreduced group scheme and $R_{a,b}$ is still a ring of invariants of a $\mu_r$-action.)

Note that $R_{1,b}$ contains the functions $u := x^r, v := y^r, w := x^{r-b}y$. This easily shows that, if $(b,r) = 1$, then $R_{1,b}$ is the normalization of the hypersurface ring

$$k[u,v,w]/\big(w^r - u^{r-b}v\big).$$

The latter ring is normal iff $b = r-1$.

THEOREM 82. *Let $k$ be an algebraically closed field and $(y \in Y)$ a normal surface singularity over $k$ whose dual graph is*

$$c_1 \;-\; c_2 \;-\; \cdots \;-\; c_{n-1} \;-\; c_n.$$

*Then an étale neighborhood of $(y \in Y)$ is isomorphic to*

$$\mathbb{A}^2/\frac{1}{D_n}\big(1, D_{n-1}\big)$$

*where the $D_r$ are computed as in (83).*

DEFINITION 83. Given a sequence $c_1, c_2, \dots$, let $M_r = M_r(c_1, \dots, c_r)$ denote the intersection form of the dual graph

$$c_1 \;-\; c_2 \;-\; \cdots \;-\; c_{r-1} \;-\; c_r.$$

For instance, for $r = 5$ we get

$$M_5(c_1, \dots, c_5) = \begin{pmatrix} c_1 & -1 & 0 & 0 & 0 \\ -1 & c_2 & -1 & 0 & 0 \\ 0 & -1 & c_3 & -1 & 0 \\ 0 & 0 & -1 & c_4 & -1 \\ 0 & 0 & 0 & -1 & c_5 \end{pmatrix}$$

Set $D_0 := 1$ and $D_r(c_1, \dots, c_r) := \det M_r(c_1, \dots, c_r)$ for $r \geq 1$. Expanding by the last column we see that the $D_r$ satisfy the recursions relation

$$D_r = c_r D_{r-1} - D_{r-2}. \tag{83.1}$$

These in turn imply that

$$M_r \cdot (D_0, D_1, \dots, D_{r-1})^t = (0, \dots, 0, D_r)^t. \tag{83.2}$$

By induction one also sees that $\gcd(D_r, D_{r-1}) = 1$ for every $r$.

It is easy to see by induction that the $D_n$ are also computed by the following continued fraction

$$\frac{D_r}{D_{r-1}} = c_r - \cfrac{1}{c_{r-1} - \cfrac{1}{c_{r-2} - \cfrac{1}{\ddots \, c_3 - \cfrac{1}{c_2 - \cfrac{1}{c_1}}}}} \tag{83.3}$$

This makes it possible to enumerate all chains with a given determinant $d$. For any $0 < e < d$ with $(d, e) = 1$, compute the above continued fraction expansion of $d/e$. These give all such chains.

For instance, the cases $\det(\Gamma) \in \{2, 3, 4, 5, 6\}$ give the possibilities

$$\begin{array}{lcl}
\det(\Gamma) = 2 & \Leftrightarrow & \Gamma \text{ is } 2 \\
\det(\Gamma) = 3 & \Leftrightarrow & \Gamma \text{ is } 3 \text{ or } 2 - 2, \\
\det(\Gamma) = 4 & \Leftrightarrow & \Gamma \text{ is } 4 \text{ or } 2 - 2 - 2, \\
\det(\Gamma) = 5 & \Leftrightarrow & \Gamma \text{ is } 5 \text{ or } 2 - 2 - 2 - 2 \text{ or } 2 - 3 \text{ or } 3 - 2, \\
\det(\Gamma) = 6 & \Leftrightarrow & \Gamma \text{ is } 6 \text{ or } 2 - 2 - 2 - 2 - 2.
\end{array}$$

84 (Proof of (82)). Let $f : X \to Y$ be the minimal resolution and $C_1, dots, C_n \subset X$ the exceptional curves. As in (83), set

$$D_r := \det M_r(c_1, \dots, c_r) \quad \text{and} \quad D_r^* := \det M_r(c_n, \dots, c_{n+1-r}).$$

By working in a suitable étale neighborhood, we can pick (nonproper) curves $C_0, C_{n+1}$ such that $C_0$ intersects only $C_1$ and $C_{n+1}$ intersects only $C_n$, both with multiplicity 1. Set $D_{-1} = D_{-1}^* = 0$. By (83.2), both of the line bundles

$$\mathcal{O}_X\bigl(-\textstyle\sum_{i=0}^{n+1} D_{i-1} C_i\bigr) \quad \text{and} \quad \mathcal{O}_X\bigl(-\textstyle\sum_{i=0}^{n+1} D_{n-i}^* C_i\bigr)$$

have degree zero on the exceptional curves $C_1, \dots, C_n$. Hence, by (76), in a neighborhood of $\mathrm{Ex}(f)$ they are both trivial. Thus, as subsheaves on $\mathcal{O}_X$, they are generated by functions $v$ (resp. $u$) such that

$$(v) = \textstyle\sum_{i=0}^{n+1} D_{i-1} C_i \quad \text{and} \quad (u) = \textstyle\sum_{i=0}^{n+1} D_{n-i}^* C_i.$$

Set $b := D_n - D_{n-1}$ and consider

$$(u^b v) = \textstyle\sum_{i=0}^{n+1} B_{i-1} C_i \quad \text{where} \quad B_{i-1} = \bigl(D_{i-1} + b D_{n-i}^*\bigr).$$

Note that the two highest coefficients are $B_n = D_n + bD_{-1} = D_n$ and $B_{n-1} = D_{n-1} + bD_0 = D_n$. Using (83.1) for both sequences, the recursion

$$B_{r-2} = c_r B_{r-1} - B_r$$

shows that all the coefficients in $(u^b v)$ are divisible by $D_n$. Thus there is a function $w$ such that

$$(w) = \sum_{i=0}^{n+1} \frac{B_{i-1}}{D_n} C_i.$$

Since $w^{D_n}$ and $u^b v$ have the same divisors, the functions $u, v, w$ are related by an equation $w^{D_n} = (\text{invertible}) u^b v$. We can set $v' := (\text{invertible}) v$ to obtain the simpler equation $w^{D_n} = u^b v'$. Thus we obtain a morphism

$$(u, v', w) : Y \to \operatorname{Spec} k[u, v', w]/(w^{D_n} = u^b v')$$

which then factors through the normalization

$$\pi : Y \to \mathbb{A}^2 / \frac{1}{D_n}\big(1, D_{n-1}\big).$$

We are left to prove that $\pi$ is étale at $y$. Since $k[u, v', w]/(w^{D_n} = u^b v')$ has degree $D_n$ over $k[u, v']$, it is enogh to prove that $(u, v') : Y \to \operatorname{Spec} k[u, v']$ also has degree $D_n$ at $y$. Equivalently, that the intersection number of their divisors

$$(u) \cdot (v) = \big(\textstyle\sum_{i=0}^{n+1} D_{i-1} C_i\big) \cdot \big(\sum_{i=0}^{n+1} D^*_{n-i} C_i\big) = 1.$$

As we noted, $C_{n+1}$ intersects only $C_n$ and $(v)$ has 0 intersection with the curves $C_1, \dots, C_n$. Thus

$$\begin{array}{rcl} \big(\sum_{i=0}^{n+1} D_{i-1} C_i\big) \cdot \big(\sum_{i=0}^{n+1} D^*_{n-i} C_i\big) & = & D_n \Big( C_{n+1} \cdot \sum_{i=0}^{n+1} D^*_{n-i} C_i \Big) \\ & = & D_n \big( C_{n+1} \cdot C_n \big) = D_n. \end{array}$$

This completes the proof of (82). □

Cyclic quotients also play a key role in the general adjunction formula.

85 (Adjunction). Let $S$ be a normal surface and $B \subset S$ a curve such that $(S, B)$ is lc. By (72), $B$ is a nodal curve and by (80), there are three types of singularities of the pair $(S, B)$:

(1) (Cyclic, plt) As in (73.2), $p \in B$ is smooth and $S$ has a cyclic quotient singularity at $p$. This is the only case when $(S, B)$ is plt at $p$.
(2) (Cyclic, lc) As in (73.1), $p \in B$ is a node and $S$ has a cyclic quotient singularity at $p$.
(3) (Dihedral) As in (73.3), $B$ is smooth at $p$ and $S$ has a dihedral quotient singularity at $p$.

Let $f : T \to S$ be a minimal log resolution of $(S, B)$ as in (64) and $B_T \subset T$ the birational transform of $B$. Write

$$f^*(K_S + B) \sim_{\mathbb{Q}} K_T + B_T + \textstyle\sum_p \Delta_p,$$

where $\Delta_p$ is supported on $f^{-1}(p)$. By the projection formula,

$$\begin{array}{rcl} \big((K_S + B) \cdot B\big) & = & \big(f^*(K_S + B) \cdot B_T\big) \\ & = & \big((K_T + B_T + \sum_p \Delta_p) \cdot B_T\big) \\ & = & 2p_a(B_T) - 2 + \sum_p \big(\Delta_p \cdot B_T\big). \end{array}$$

For the cases (85.2–3), the divisor $\Delta_p$ was computed in (73). For both of these, the curves intersecting $B_T$ have coefficient 1 in $\Delta_p$. Next we compute the discrepancies for (85.1).

*Claim* 85.4. For the singularity $\bullet - c_1 - \cdots - c_n$ the discrepancy divisor is

$$\Delta = \sum_{i=1}^{n} \Big(1 - \frac{D_{i-1}}{\det(\Gamma)}\Big) C_i.$$

In particular, $B \cdot \Delta = 1 - \frac{1}{\det(\Gamma)}$.

Proof. By the adjunction formula,

$$\big((K+B+C_1+\cdots+C_n)\cdot C_j\big)=\begin{cases}0 & \text{if } j<n, \text{ and}\\ 1 & \text{if } j=n.\end{cases}$$

Thus, using (83.2), we conclude that

$$\Big(\Big(K+B+C_1+\cdots+C_n-\frac{1}{\det(\Gamma)}\sum_{i=1}^{n}D_{i-i}C_i\Big)\cdot C_j\Big)=0\quad\forall i.$$

This gives the fiormula for $\Delta$. The only curve in $\Delta$ that intersects $B$ is $C_1$ and its coefficients is $1-\frac{D_0}{\det(\Gamma)}=1-\frac{1}{\det(\Gamma)}$. □

Putting these together, we obtain that

$$\begin{aligned}\big((K_S+B)\cdot B\big) &= 2p_a(B_T)-2+\textstyle\sum_{\text{cyclic, plt}}\Big(1-\frac{1}{\det(\Gamma_p)}\Big)+\\ &\quad +2\cdot\#\{\text{cyclic lc points}\}+\#\{\text{dihedral points}\}.\end{aligned}$$

Since $B_T$ is the normalization of $B$, $p_a(B_T)=p_a(B)-\#\{\text{nodes}\}$ and the nodes of $B$ correspond to the cyclic lc points. Thus we have proved the following general adjunction formula:

THEOREM 86 (Adjunction formula). *Let $S$ be a normal surface and $B\subset S$ a curve such that $(S,B)$ is lc. Then*

$$\big((K_S+B)\cdot B\big)=2p_a(B)-2+\sum_{p:\ \text{cyclic and plt}}\Big(1-\frac{1}{\det(\Gamma_p)}\Big)+\#\{\textit{dihedral points}\}.\quad\square$$

87 (Different I.). As stated, (86) is only an equality of two numbers. However, by looking at its proof, we get a canonical isomorphism of two sheaves. If $U\subset S$ is an open subset such that $U$ and $B\cap U$ are both smooth then taking the Poincaré residue gives a canonical isomorphism

$$\omega_U(B)|_{U\cap B}\cong\omega_{U\cap B}. \tag{87.1}$$

(We discuss this in greater detail in (121).) Next we study this isomorphism near the singular points.

Choose $m>0$ such that $mK_S$ and $mB$ are both Cartier. Then $\omega_S^{[m]}(mB)|_B$ is a line bundle and the proof in (85) shows that there is well-defined $\mathbb{Q}$-divisor $\operatorname{Diff}_B$ called the *different*, supported on the points where $S$ is singuar, such that the $m$th power of (87.1) extends to an isomorphism

$$\omega_S^{[m]}(mB)|_B\cong\omega_B^m\big(m\operatorname{Diff}_B\big). \tag{87.2}$$

We can restate and refine (86) as a computation of the different:

$$\operatorname{Diff}_B=\sum_{p:\ \text{cyclic and plt}}\Big(1-\frac{1}{\det(\Gamma(p))}\Big)[p]+\sum_{p:\ \text{dihedral}}[p]. \tag{87.3}$$

For later purposes, we also study the different when $(S,B)$ is not log canonical and we take into account the influence of another effecive $\mathbb{Q}$-divisor $B'$ which has no components in common with $B$. Since the singularities of $B$ can be now quite complicated, we compute everything on the normalization $\bar B\to B$.

Going through the above arguments again, if $mK_S+mB+mB'$ is a Cartier divisor, we obtain a different $\operatorname{Diff}_{\bar B}(B')$ such that

$$\omega_S^{[m]}(mB+mB')|_{\bar B}\cong\omega_{\bar B}^m\big(m\operatorname{Diff}_{\bar B}(B')\big). \tag{87.4}$$

We can not give a precise formula for $\operatorname{Diff}_{\bar{B}}(B')$, but we can compare it to the discrepancies $a(C, S, B+B')$ of exceptional curves.

*Claim* 87.5. Let $S$ be a normal surface and $B \subset S$ a reduced curve with normalization $\bar{B} \to B$. For a point $p \in B$, let $\bar{p} \in \bar{B}$ denote any preimage. Let $B'$ be an effecive $\mathbb{Q}$-divisor that has no components in common with $B$ and assume that $m(K_S + B + B')$ is a Cartier divisor for some $m > 0$.

Let $f : T \to S$ be a log resolution of $(S, B + B')$ and, as in (64), write $K_T + B_T + B'_T + \Delta \sim_{\mathbb{Q}} f^*(K_S + B + B')$ where $B_T, B'_T \subset T$ are the birational transforms of $B, B'$. Note that $B_T = \bar{B}$. Define the different as

$$\operatorname{Diff}_{\bar{B}}(B') := (B'_T + \Delta)|_{\bar{B}}.$$

Then $\operatorname{Diff}_{\bar{B}}(B')$ is independent of the choice of $f$. Furthermore:

i) If $(S, B+B')$ is lc at $p$, then the coefficient of $[\bar{p}]$ in the different $\operatorname{Diff}_{\bar{B}}(B')$ is the negative of the smallest discrepancy of $(S, B + B')$ above $p$.

ii) If $(S, B + B')$ is not lc at $p$ then the coefficient of $[\bar{p}]$ in the different $\operatorname{Diff}_{\bar{B}}(B')$ is greater than 1.

A far reaching generalization of this will be given in Section 4.

Proof. By the usual adjunction formula,

$$f^*(K_S + B + B')|_{\bar{B}} - K_{B_T} = \big(B'_T + \Delta\big)|_{B_T} = \operatorname{Diff}_{\bar{B}}(B').$$

Hence $\operatorname{Diff}_{\bar{B}}(B')$ is independent of the choice of $f$ and the coefficient of $[\bar{p}]$ in $\operatorname{Diff}_{\bar{B}}(B')$ equals the local intersection number $\big(B_T \cdot B'_T + \Delta\big)_{\bar{p}}$.

Write $\Delta = \sum d_i C_i$, where $d_i \geq 0$ by (66.3). Assume that $(S, B + B')$ is not log terminal at $p \in S$, that is, $d_r \geq 1$ for some $C_r \subset f^{-1}(p)$.

Choose a chain of exceptional curves $C_1, \dots, C_{r-1} \subset f^{-1}(p)$ such that $B_T$ intersects $C_1$ at $\bar{p}$ and $C_i$ intersects $C_{i+1}$ for $i = 1, \dots, r-1$. Write

$$K_T + B_T + B'_T + \Delta = K_T + \textstyle\sum_{i=1}^{r-1} d_i C_i + \Delta',$$

and compute the intersection numbers

$$\begin{aligned}
&\Big(\big(\textstyle\sum_{i=1}^{r-1}(d_i - 1)C_i\big) \cdot C_j\Big) = \\
&\quad = -\big(K_T + \Delta' \cdot C_j\big) - \Big(\big(\textstyle\sum_{i=1}^{r-1} C_i\big) \cdot C_j\Big) \\
&\quad = -\Big(\big(K_T + C_j\big) \cdot C_j\Big) - \Big(\big(\Delta' + \textstyle\sum_{i \neq j} C_i\big) \cdot C_j\Big) \\
&\quad = -2p_a(C_j) + 2 - \Big(\big(\Delta' + \textstyle\sum_{i \neq j} C_i\big) \cdot C_j\Big).
\end{aligned}$$

If $2 \leq j \leq r-2$, then at least 2 of the other $C_i$ intersect $C_j$, hence the last term is $\geq 2$. If $j = 1$ or $j = r-1$, then at least 1 of the other $C_i$ intersect $C_j$ and either $B_T$ or $d_r C_r$ intersect $C_j$ with multiplicity $\geq 1$. Thus

$$\Big(\big(\textstyle\sum_{i=1}^{r-1}(d_i - 1)C_i\big) \cdot C_j\Big) \leq 0 \quad \text{for } j = 1, \dots, r-1,$$

hence, by (79.1), $\sum_{i=1}^{r-1}(d_i - 1)C_i$ is a semi positive vector and positive if $d_r > 1$. Since

$$\big(B_T \cdot B'_T + \Delta\big)_{\bar{p}} \geq \big(B_T \cdot d_1 C_1\big)_{\bar{p}} \geq d_1,$$

we have proved (ii) and also (i) in case $(S, B + B')$ is not log terminal.

If $(S, B + B')$ is log terminal then so is $(S, B)$ and we are in case (85.1).

Write $\Delta = \sum_i (1 - \epsilon_i) C_i$ and drop $p$ from the notation temporarily. It is convenient to set $\epsilon_0 = 0$ and $\epsilon_{n+1} = 1$.

We can rewrite $\big(C_i\cdot(K_T+B_T+B'_T+\Delta)\big)=0$ as

$$-2+\epsilon_i c_i+1-\epsilon_{i-1}+1-\epsilon_{i+1}+(C_i\cdot B')=0 \quad \text{for } i=1,\dots,n.$$

The latter can be rearranged to

$$\epsilon_i=\frac{\epsilon_{i-1}+\epsilon_{i+1}}{c_i}-\frac{(C_i\cdot B'_p)}{c_i}\le\frac{\epsilon_{i-1}+\epsilon_{i+1}}{c_i}.$$

Since $c_i\ge 2$, we conclude that $\epsilon_i$ is a convex function of $i$. Since $\epsilon_0=0$ we see that either $\epsilon_1<0$ or $0\le\epsilon_1\le\epsilon_2\le\cdots\le\epsilon_n$.

In the first case, $(S,B+B')$ is not log canonical. Otherwise, $a(C_1,S,B+B')=-1+\epsilon_1$ is the smallest disrepancy and the coefficient of $[\bar p]$ in $\operatorname{Diff}_{\bar B}(B')$ equals the local intersection number

$$\big(B_T\cdot B'_T+\Delta\big)_{\bar p}=\big(B_T\cdot(1-\epsilon_1)C_1\big)=1-\epsilon_1. \quad \square$$

**Other quotients.**

88. Let $(\Gamma,B)$ be a tree with 1 fork where the components of $B$ intersect only the leaves. That is, the dual graph is

$$\begin{array}{ccccccccc} B_1 & - & \Gamma_1 & - & c_0 & - & \Gamma_2 & - & B_2\\ &&&& | &&&&\\ &&&& \Gamma_3 &&&&\\ &&&& | &&&&\\ &&&& B_3 &&&& \end{array} \tag{88.1}$$

The fork is denoted by $C_0$ and, for $i=1,2,3$, the curves on the branches are indexed as

$$\Big(C_0\ -\ \Gamma_i\ -\ B_i\Big)\ =\ \Big(C_0\ -\ C^i_1\ -\cdots-\ C^i_{n(i)}\ -\ B_i\Big).$$

We allow the degenerate case when some $\Gamma_i$ is empty. The curves $C^i_{n(i)}$ are called *leaves* and the curves $C^i_j$ for $0<j<n(i)$ are called intermediate. Set $c^i_j=-(C^i_j\cdot C^i_j)$ and $\beta_i=(C^i_{n(i)}\cdot B_i)$.

THEOREM 89. *Notation as above. Then (88.1) is numerically log canonical iff*

$$\frac{1-\beta_1}{\det(\Gamma_1)}+\frac{1-\beta_2}{\det(\Gamma_2)}+\frac{1-\beta_3}{\det(\Gamma_3)}\ge 1. \tag{89.1}$$

Proof. Let $\Sigma^i:=\sum_j C^i_j$ denote the sum of all curves in $\Gamma_i$ with coefficient 1 and $\Sigma_\Gamma:=C_0+\sum_i\Sigma^i$ the sum of all curves in $\Gamma$.

Write the discrepancy divisor as $\Delta=a_0C_0+\sum_i\Delta_i$ where $\Delta_i$ involves only the curves in $\Gamma_i$. Thus $\Sigma_\Gamma-\Delta\sim_{\mathbb Q}K_\Gamma+\Sigma_\Gamma+B$ and for every exceptional curve $C$ we have

$$\big((\Sigma_\Gamma-\Delta)\cdot C\big)=-2+\#\{\text{neighbors of } C\}+(B\cdot C).$$

In particular,

$$\big((\Sigma_\Gamma-\Delta)\cdot C\big)=\begin{cases}1 & \text{if } C=C_0 \text{ is the fork}\\ 0 & \text{if } C=C^i_j \text{ is intermediate, and}\\ \beta_i-1 & \text{if } C=C^i_{n(i)} \text{ is a leaf.}\end{cases}$$

Using $D^i_j=\det M(b^i_1,\dots,b^i_j)$, consider the $\mathbb Q$-divisor

$$\mathbf{D}^i:=\frac{1}{\det(\Gamma_i)}\big(D^i_0C_1+\cdots+D^i_{n(i)-1}C_{n(i)}\big),$$

where $\det(\Gamma_i) = D^i_{n(i)}$ is the determinant of $\Gamma_i$. Therefore, by (83.2),

$$\Big((\Sigma_\Gamma - \Delta - \sum_{i=1}^{3}(1-\beta_i)\mathbf{D}^i)\cdot C\Big) = \begin{cases} 1 - \sum_{i=1}^{3}\frac{1-\beta_i}{\det(\Gamma_i)} & \text{if } C \text{ is the fork} \\ 0 \quad \text{if } C \text{ is any other curve.} \end{cases} \tag{89.2}$$

Thus, if (89.1) holds then $\Sigma_\Gamma - \Delta - \sum_{i=1}^{3}(1-\beta_i)\mathbf{D}^i$ has $\leq 0$ intersection with every curve. Hence, by (79.1), it is a semipositive vector. Therefore

$$\Delta \leq \Sigma_\Gamma - \sum_{i=1}^{3}(1-\beta_i)\mathbf{D}^i \leq \Sigma_\Gamma,$$

showing that $(\Gamma, B)$ is numerically lc.

If (89.1) fails then $\Sigma_\Gamma - \Delta - \sum_{i=1}^{3}(1-\beta_i)\mathbf{D}^i$ is a negative vector. Thus $C_0$ appears in $\Delta$ with coeffienect $> 1$, hence $(\Gamma, B)$ is not numerically lc. □

**List of log canonical surface singularities.**

Here is a short summary of the results proved so far.

90 (Log canonical, not log terminal).

(90.1) (Simple elliptic) $\Gamma = \{E\}$ has a single vertex which is a smooth elliptic curve with self intersection $\leq -1$. $B$ has to be 0.

(90.2) (Cusp) $\Gamma$ is a circle of smooth rational curves, $c_i \geq 2$ and at least one of them with with $c_i \geq 3$. The cases $n = 1, 2$ are somewhat special. $B$ has to be 0.

$$\begin{array}{ccccccc} & & c_n & - \cdots - & c_{r+1} & & \\ & / & & & & \backslash & \\ c_1 & & & & & & c_r \\ & \backslash & & & & / & \\ & & c_2 & - \cdots - & c_{r-1} & & \end{array}$$

(90.3) ($\mathbb{Z}/2$-quotient of a cusp or simple elliptic) $\Gamma$ has 2 forks and each branch is a single $(-2)$-curve. We have $c_i \geq 2$ and $B$ has to be 0.

$$\begin{array}{ccccccc} 2 & & & & & & 2 \\ & \backslash & & & & / & \\ & & c_1 & - \cdots - & c_n & & \\ & / & & & & \backslash & \\ 2 & & & & & & 2 \end{array}$$

(90.4) (Other quotients of a simple elliptic) The dual graph is

$$\begin{array}{ccccc} \Gamma_1 & - & c_0 & - & \Gamma_2 \\ & & | & & \\ & & \Gamma_3 & & \end{array}$$

with 3 possibilities for $\big(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3)\big)$:

($\mathbb{Z}/3$-quotient) (3,3,3)
($\mathbb{Z}/4$-quotient) (2,4,4)
($\mathbb{Z}/6$-quotient) (2,3,6).

In all cases, $B$ has to be 0.

91 (Log terminal). Here $B$ can be nonzero.

(91.1) (Cyclic quotient)

$$c_1 - \cdots - c_n$$

(91.2) (Dihedral quotient) Here $n \geq 2$ with dual graph

$$\begin{array}{ccccccc} & & & & & & 2 \\ & & & & & \diagup & \\ c_1 & - & \cdots & - & c_n & & \\ & & & & & \diagdown & \\ & & & & & & 2 \end{array}$$

(91.3) (Other quotients) The dual graph is as in (90.4) whith 3 cases for $\big(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3)\big)$:

(Tetrahedral) (2,3,3)
(Octahedral) (2,3,4)
(Icosahedral) (2,3,5).

**Boundary with coefficients $\geq \frac{1}{2}$.**

Here we classify all log canonical surface pairs $(S, B)$ where $B = \sum \beta_i B_i$ and $\beta_i \in \{1, \frac{1}{2}\}$ for every $i$.

This method essentially also classifies all pairs $(S, B)$ such that $\beta_i \geq \frac{1}{2}$ for at least one value of $i$. Indeed, in this case we can replace $(S, \sum \beta_i B_i)$ by $(S, \sum \beta_i^* B_i)$ where $\beta_i^* = \frac{1}{2}\lfloor 2\beta_i \rfloor$ is the "half integral round down."

The main remaining interesting issue is the following. Given a lc pair $(S, \sum \beta_i^* B_i)$, for which values of $\beta_i \geq \beta_i^*$ is $(S, \sum \beta_i B_i)$ also log canonical. In almost all instances our formulas give the answer, but some case analysis is left undone.

As before, we consider the dual graph of the log minimal resolution, and we show that the $B_i$ can be replaced by $(-2)$-curves such that the resulting new dual graph is still lc. We then read off the classification of the pairs $(S, B)$ from the classification of all lc dual graphs.

92 (Problems with small coefficients). Let $S$ be a smooth or log terminal surface and $C$ an arbitrary effective curve on $S$. It is easy to see from the definition that $(S, \epsilon C)$ is klt for all $0 \leq \epsilon \ll 1$. Thus a classification of all possible lc pairs $(S, B)$ would include a classification of all curve singularities on smooth or log terminal surfaces. This is an interesting topic on its own right but not our main concern here.

For a pair $(S, C)$, the optimal bound on $\epsilon$ depends on $S$ and $C$ in a quite subtle way. See [**KSC04**, Sec.6.5] for the case when $S$ is smooth. The general case has not yet been worked out.

For several reasons, the most important fractional coefficients are 1 and $1 - \frac{1}{n}$. These are all $\geq \frac{1}{2}$, exactly the value where our classification works.

93 (Replacing $B$ by $(-2)$-curves). Let $(\Gamma, B)$ be an extended lc dual graph. Write $B = B' + \sum_j \frac{1}{2} B_j$ where we do not assume that the $B_j$ are distinct.

Usually $B_j$ intersects exactly one curve $C_{a(j)}$ and the intersection is transverse. However, $B_j$ may intersect $\sum C_i$ at an intersection point or it may be tangent to some $C_i$. Keeping these in mind, we construct a new dual graph $(\Gamma^*, B')$ as follows.

For each $i$, set $d(i) := (C_i \cdot \sum_j B_j)$ and introduce $d(i)$ new vertices $C_{i1}, \dots, C_{i,d(i)}$. The intersection numbers are unchanged for the old vertices, $c_{ij} = -(C_{ij}^2) = 2$ and $(C_i \cdot C_{ij}) = 1$. No other new intersections.

93.1 *Claim.* If $(\Gamma, B)$ is lc then $(\Gamma^*, B')$ is also lc.

Proof. Let $\Delta = \sum_{i=1}^{n} d_i C_i$ be the discrepancy divisor. Thus

$$(\Delta \cdot C_i) = 2 - c_i - (C_i \cdot B) = 2 - c_i - \tfrac{1}{2}d(i) - (C_i \cdot B').$$

Set $\Delta^* = \sum_{i=1}^{n} d_i C_i + \frac{1}{2}\sum C_{ij}$. Then

$$\big((\Delta^* + B') \cdot C\big) = \begin{cases} 2 - c_i - (C_i \cdot B) + \frac{1}{2}d(i) = 2 - c_i - (C_i \cdot B') & \text{if } C = C_i, \text{ and} \\ d_i - \frac{1}{2}(C_{ij}^2) \leq 0 = 2 - c_{ij} & \text{if } C = C_{ij}. \end{cases}$$

Thus, by (79.1), $\Delta(\Gamma^*, B') \leq \Delta^*$ and so $(\Gamma^*, B')$ is also lc. □

If $B$ is integral, then we attach at least two $(-2)$-curves to $C_i$. There are very few cases when a lc graph contains two $(-2)$-leaves attached to the same curve. This establishes the completeness of the list in (73).

94 (Log canonical, $B$ contains a half integral divisor).
We classify these by the following scheme:

1. (Two fork case) This happens when
   - (a) $\Gamma$ is not cyclic and there is a $\frac{1}{2}B_i \leq B$ which intersects a $C$ which is not a leaf.
   - (b) $\Gamma$ is cyclic and there are $\frac{1}{2}B_i + \frac{1}{2}B_j \leq B$ which intersect a $C$ which is not a leaf.
   - (c) $\Gamma$ is cyclic and there is a $\frac{1}{2}B_i \leq B$ which intersects a $C$ which is not a leaf with multiplicity 2.
2. (One fork case)
   - (a) $\Gamma$ is not cyclic and every $\frac{1}{2}B_i \leq B$ intersect a leaf.
   - (b) $\Gamma$ is cyclic and there is only one $\frac{1}{2}B_i \leq B$ that intersects a $C$ which is not a leaf.
3. (Triple intersection point case) There is a $\frac{1}{2}B_i \leq B$ which intersects $\Gamma$ at a singular point.

(94.1) (Two fork case) Either all $c_i \geq 2$ or $n = 1$ and $c_1 = 1$ with dual graph

$$\begin{array}{ccccccccc} \circledast & & & & & & & & \circledast \\ & \diagdown & & & & & & \diagup & \\ & & c_1 & - & \cdots & - & c_n & & \\ & \diagup & & & & & & \diagdown & \\ \circledast & & & & & & & & \circledast \end{array}$$

where $\circledast$ can be a $(-2)$-curve, a component of $B$ with coefficient $\geq \frac{1}{2}$ or it can be missing entirely.

An interesting special case is

$$\begin{array}{ccccc} 2 & - & c_1 & - & 3 \\ & & | & & \\ & & B_i & & \end{array}$$

For $c_1 = 1$ this is the minimal log resolution of the planar cusp. In all cases, the lc threshold is $5/6$, that is $(S, \frac{5}{6}B)$ is lc.

If there is a $\frac{1}{2}B_i \leq B$ which intersects a $C$ which is not a leaf with multiplicity 2, we get the special case

$$\begin{array}{ccccc} 2 & - & c_1 & - & 2 \\ & & \| & & \\ & & B_i & & \end{array}$$

(94.2) (One fork case) In all other cases without triple intersections, the dual graph is

$$\begin{array}{ccccccccc} B_1 & - & \Gamma_1 & - & c_0 & - & \Gamma_2 & - & B_2 \\ & & & & | & & & & \\ & & & & \Gamma_2 & - & B_3 & & \end{array}$$

where $B_i$ appears in $B$ with coefficient $\beta_i$ which is either $\frac{1}{2}$ or 0. We allow the degenerate possibility that some $\Gamma_i$ is empty.

If at least 2 of the $\Gamma_i$ are empty, then we are in a special instance of the two fork case.

Next we enumerate the other possibilities.

From (89), we get the log canonicity condition

$$\frac{1-\beta_1}{\det(\Gamma_1)} + \frac{1-\beta_2}{\det(\Gamma_2)} + \frac{1-\beta_3}{\det(\Gamma_3)} \geq 1.$$

(3 components) Here all $\beta_i = \frac{1}{2}$ and so we get

$$\frac{1}{\det(\Gamma_1)} + \frac{1}{\det(\Gamma_2)} + \frac{1}{\det(\Gamma_3)} \geq 2.$$

This is only possible if one of the $\det(\Gamma_i)$ is 1 and the others are 2. This gives the dual graph

$$\begin{array}{ccccccccc} \circledast & - & 2 & - & c_0 & - & 2 & - & \circledast \\ & & & & | & & & & \\ & & & & \circledast & & & & \end{array}$$

(2 components) The condition is

$$\frac{1}{\det(\Gamma_1)} + \frac{2}{\det(\Gamma_2)} + \frac{1}{\det(\Gamma_3)} \geq 2,$$

where $\det(\Gamma_2) \geq 2$ since otherwise we are in the two fork case.

This gives the dual graph

$$\begin{array}{ccccccccc} \circledast & - & 2 & - & c_0 & - & 2 & - & \circledast \\ & & & & | & & & & \\ & & & & 2 & & & & \end{array}$$

(1 component) The condition is

$$\frac{2}{\det(\Gamma_1)} + \frac{2}{\det(\Gamma_2)} + \frac{1}{\det(\Gamma_3)} \geq 2,$$

where $\det(\Gamma_1), \det(\Gamma_2) \geq 2$. If $\det(\Gamma_1) = \det(\Gamma_2) = 2$ then we are in the two fork case.

Otherwise, the only possibilities are

$$\begin{array}{ccccccc} \Gamma_1 & - & c_0 & - & 2 & - & \circledast \\ & & | & & & & \\ & & 2 & & & & \end{array} \qquad \text{with } \det(\Gamma_1) \in \{3, 4\}, \text{ and}$$

$$\begin{array}{ccccccc} \Gamma_1 & - & c_0 & - & \Gamma_3 & - & \circledast \\ & & | & & & & \\ & & 2 & & & & \end{array} \qquad \text{with } \det(\Gamma_1) = \det(\Gamma_3) = 3.$$

(94.3) (Triple intersection cases)

These are the cases when $B$ passes through the intersection point of 2 curves $C_j$. The procedure of (93) tells us to attach $(-2)$-curves to both of these $C_j$. This is a strong restriction, and all such cases have been enumerated above. We just need to look at the cases when the components of $B$ are attached to a pair of intersecting curves. This can happen in only a few cases.

A separate treatment of these possibilities is needed, however, since not all the remaining cases are lc. We have encountered a similar problem in (77). All curves in the dual graph have discrepancy $\geq -1$ but a further blow up may show that $(S, B)$ is not lc. These cases were enumerated by hand and we get only the following possibilities

$$\begin{array}{ccc} 2 & - & b \\ & | & \\ & \circledast & \end{array} \quad \text{with } 2 \leq b \leq 5, \qquad \begin{array}{ccc} 3 & - & 3 \\ & | & \\ & \circledast & \end{array} \qquad \text{or} \qquad \begin{array}{ccccc} 2 & - & 2 & - & 2 \\ & & & | & \\ & & & \circledast & \end{array}$$

**Auxiliary results.**

Here we prove three well know results on birational maps of surfaces: the Hodge index theorem, the Grauert-Riemenschneider vanishing theorem and Castelnuovo's contraction theorem. Instead of the usual setting, we consider these for excellent 2-dimensional schemes.

First we prove the Hodge index theorem. That is, we show that the intersection matrix of the exceptional curves of a proper morphism of surfaces is negative definite.

THEOREM 95 (Hodge index theorem). *Let $X$ be a 2-dimensional regular scheme, $Y$ an affine scheme and $f : X \to Y$ a proper and generically finite morphism with exceptional curves $\cup C_i$. Then the intersection form $(C_i \cdot C_j)$ is negative-definite.*

Proof. It is enough to consider all the exceptional curves that lie over a given $y \in Y$. Then all the curves considered are proper over the residue field $k(y)$, so the intersection numbers

$$(C_i \cdot C_j) := \deg_{k(y)} \mathcal{O}_X(C_i)|_{C_j}$$

are defined as usual. Note the obvious property that

$$(C_i \cdot C_j) \geq 0 \quad \text{for } i \neq j. \tag{95.1}$$

By (96), there is an effective $f$-exceptional Cartier divisor. An easy argument (97.4), using only (95.1) and bilinear algebra, shows that the intersection form is negative-definite. □

LEMMA 96. *Let $f : X \to Y$ be as in (95). Then $f$ is projective and there is an effective $f$-exceptional Cartier divisor $W$ on $X$ such that $-W$ is $f$-ample.*

Proof. For each exceptional curve $C_i \subset X$ let $U_i \subset X$ be an open affine subset such that $C_i \cap U_i \neq \emptyset$. Let $H_i \subset U_i$ be an effective divisor which intersects $C_i$ such that $C_i \not\subset H_i$ and let $\bar{H}_i \subset X$ denote its closure. Then $H := \sum \bar{H}_i$ has positive intersection number with every exceptional curve, hence it is $f$-ample. Its push-forward $f(H) \subset Y$ is an effective Weil divisor on $Y$; it is thus contained in an effective Cartier divisor $D$ since $Y$ is affine. Write $f^*(D) = H + D' + W$ where $W$ is effective, $f$-exceptional and $D'$ is effective without $f$-exceptional irreducible components. Then

$$(W \cdot C_i) = (f^*D \cdot C_i) - (H \cdot C_i) - (D' \cdot C_i) \leq -(H \cdot C_i) < 0,$$

so $-W$ is $f$-ample. □

97 (Remarks on quadratic forms). Fix a basis $C_1, \dots, C_n$ in a real vector space $V$. Let $B(\ ,\ )$ be a bilinear form on $V$. We usually write $(C_i \cdot C_j)$ instead of $B(C_i, C_j)$. We are interested in the cases that look like the intersection form of a collection of curves on a surface. That is, we impose the following:

97.1 *Assumption.* $(C_i \cdot C_j) \geq 0$ if $i \neq j$.

97.2 *Notation.* A vector $\sum a_i C_i$ is called *semipositive* if $a_i \geq 0$ for every $i$ and *positive* if $a_i > 0$ for every $i$. Given two vectors $A = \sum a_i C_i$ and $A' = \sum a'_i C_i$, write $A \leq A'$ iff $a_i \leq a'_i$ for every $i$ and $A \ll A'$ iff $a_i < a'_i$ for every $i$. Set $\operatorname{Supp}(\sum a_i C_i) := \{C_i : a_i \neq 0\}$.

We say that $B(\ ,\ )$ is *decomposable* if we can write $\{1, \dots, n\} = I \cup J$ such that $(C_i \cdot C_j) = 0$ whenever $i \in I$ and $j \in J$. Decomposable forms correspond to disconnected sets of curves.

97.3 *Lemma.* Let $B(\ ,\ )$ be an indecomposable bilinear form satisfying (97.1). Assume that there is a positive vector $W = \sum w_i C_i$ such that $(C_i \cdot W) \leq 0$ for every $i$ with strict inequality for some $i$. Let $Z$ be any vector such that $Z \not\leq 0$. Then there is curve $C_j$ such that $(C_j \cdot Z) < 0$ and $C_j$ has positive coefficient in $Z$.

Proof. Choose the maximal $a \in \mathbb{R}^+$ such that $az_i \leq w_i$ for every $i$. Then $\sum (w_i - az_i) C_i$ is semipositive; set $I := \{i : w_i - az_i > 0\}$. If $I = \emptyset$ then $Z = \frac{1}{a} W$ and $C_j$ exists by assumption. Otherwise $I \neq \emptyset$, and since $B(\ ,\ )$ is indecomposable, there is an index $j$ such that $w_j - az_j = 0$ and $(C_j \cdot C_i) > 0$ for some $i \in I$. Then

$$\left(C_j \cdot \textstyle\sum (w_i - az_i) C_i\right) \geq (w_i - az_i)(C_j \cdot C_i) > 0.$$

Thus $a(C_j \cdot Z) = (C_j \cdot W) - (C_j \cdot (W - aZ)) < 0$. □

97.4 *Claim.* Let $B(\ ,\ )$ be bilinear form satisfying (97.1).

(1) $B(\ ,\ )$ is negative definite iff there is a semipositive vector $W = \sum w_i C_i$ such that $(C_i \cdot W) < 0$ for every $i$.
(2) If $B(\ ,\ )$ is indecomposable, then it is negative definite iff there is a semipositive vector $W = \sum w_i C_i$ such that $(C_i \cdot W) \leq 0$ for every $i$ and strict inequality holds for some $i$.

Proof. Consider the function $f(x_1, \dots, x_n) = (\sum x_i C_i \cdot \sum x_i C_i)$ on any cube $0 \leq x_i \leq N$. Since

$$\frac{\partial f}{\partial x_j}(Z) = 2(C_j \cdot Z),$$

we see from (97.3) that $f$ strictly increases as we move toward the origin parallel to one of the coordinate axes. Thus $f$ has a strict maximum at the origin and so $(Z \cdot Z) < 0$ for any semipositive vector $Z \neq 0$.

Any $Z$ can be written as $Z^+ - Z^-$ where $Z^+, Z^-$ are semipositive with disjoint supports. Then

$$(Z \cdot Z) = (Z^+ \cdot Z^+) + (Z^- \cdot Z^-) - 2(Z^+ \cdot Z^-) \leq (Z^+ \cdot Z^+) + (Z^- \cdot Z^-),$$

and we have seen that the last 2 terms are $\leq 0$ with equality only if $Z = 0$.

The converse follows from the next, more general result. □

97.5 *Claim.* Let $B(\ ,\ )$ be an indecomposable negative definite bilinear form satisfying (97.1). Let $Z = \sum a_i C_i$ be a vector such that $(Z \cdot C_j) \geq 0$ for every $j$. Then

(1) either $Z = 0$ (and so $(Z \cdot C_j) = 0$ for every $j$),
(2) or $-Z$ is positive (and $(Z \cdot C_j) > 0$ for some $j$).

Proof. Write $Z = Z^+ - Z^-$ where $Z^+, Z^-$ are semipositive with disjoint supports. If $Z^+ \neq 0$ then $(Z^+ \cdot Z^+) < 0$. Hence there is a $C_i \subset \operatorname{Supp} Z^+$ such that $(C_i \cdot Z^+) < 0$. $C_i$ is not in $\operatorname{Supp} Z^-$, so $(C_i \cdot Z^-) \geq 0$ and so $(C_i \cdot Z) < 0$, a contradiction.

Finally, if $\emptyset \neq \operatorname{Supp} Z^- \neq \cup_i C_i$, then there is a curve $C_i$ such that $C_i \not\subset \operatorname{Supp} Z^-$ but $(C_i \cdot Z^-) > 0$. Then $(C_i \cdot Z) = -(C_i \cdot Z^-) < 0$, again a contradiction. □

The following is a strengthening of the Grauert-Riemenschneider vanishing theorem for surfaces, using the method of [**Lip69**].

THEOREM 98. *Let $X$ be a regular surface and $f : X \to Y$ a proper, generically finite morphism with exceptional curves $C_i$ such that $\cup_i C_i$ is connected. Let $L$ be a line bundle on $X$ and assume that there exist $\mathbb{Q}$-divisors $N$ and $\Delta_X = \sum d_i C_i$ such that*

(1) *$L \cdot C_i = (K_X + N + \Delta_X) \cdot C_i$ for every $i$,*
(2) *$N \cdot C_i \geq 0$ for every $i$, and*
(3) *$\Delta_X$ satisfies one of the following*
   (a) *$0 \leq d_i < 1$ $\forall i$,*
   (b) *$0 < d_i \leq 1$ $\forall i$ and $d_j \neq 1$ for some $j$,*
   (c) *$0 < d_i \leq 1$ $\forall i$ and $N \cdot C_j > 0$ for some $j$.*

*Then $R^1 f_* L = 0$.*

Proof. Let $Z = \sum_{i=1}^{s} r_i C_i$ be an effective integral cycle. We prove by induction on $\sum r_i$ that

$$H^1(Z, L \otimes \mathcal{O}_Z) = 0. \tag{98.4}$$

Using the Theorem on Formal Functions, this implies (98).

Let $C_i$ be an irreducible curve contained in $\operatorname{Supp} Z$. Set $Z_i = Z - C_i$ and consider the short exact sequence:

$$0 \to \mathcal{O}_{C_i} \otimes \mathcal{O}_X(-Z_i) \cong \mathcal{O}_X(-Z_i)/\mathcal{O}_X(-Z) \to \mathcal{O}_Z \to \mathcal{O}_{Z_i} \to 0.$$

Tensoring with $L$ we obtain

$$0 \to \mathcal{O}_{C_i} \otimes L(-Z_i) \to L \otimes \mathcal{O}_Z \to L \otimes \mathcal{O}_{Z_i} \to 0.$$

By induction on $\sum r_i$, $H^1(Z_i, L \otimes \mathcal{O}_{Z_i}) = 0$. Thus it is enough to prove that $H^1(C_i, \mathcal{O}_{C_i} \otimes L(-Z_i)) = 0$ for some $i$. This in turn would follow from

$$L \cdot C_i - Z_i \cdot C_i > \deg \omega_{C_i} = C_i^2 + K_X \cdot C_i,$$

which is equivalent to

$$N \cdot C_i + (\Delta_X - Z) \cdot C_i > 0.$$

By assumption, $N \cdot C_i \geq 0$ always holds.

If $Z \not\leq \Delta_X$ the we can apply (97.3) to $Z - \Delta_X$ to obtain $C_i \subset \operatorname{Supp} Z$ such that $(\Delta_X - Z) \cdot C_i > 0$. This settles case (3.a).

Thus assume that $Z \leq \Delta_X$. If $\operatorname{Supp} Z = \operatorname{Supp} \Delta_X$ then $Z = \Delta_X$. This can happen only in case (3.c), but then $N \cdot C_i > 0$ for some $i$ and we are done.

Finally, if $\operatorname{Supp} Z \neq \operatorname{Supp} \Delta_X$ then $\operatorname{Supp} Z$ and $\operatorname{Supp}(\Delta_X - Z)$ intersect in finitely many points and this intersection is nonempty in cases (3.b–c). Thus again there is a $C_i \subset \operatorname{Supp} Z$ such that $(\Delta_X - Z) \cdot C_i > 0$. □

THEOREM 99 (Castelnuovo's contractibility criterion). *Let $X$ be a regular surface, $f : X \to Y$ a projective morphism and $E \subset X$ an irreducible and reduced curve such that $(E \cdot K_X) < 0$ and $f(E)$ is 0-dimensional.*

*Then there is a proper morphism to a regular pointed surface $g : (E \subset X) \to (z \in Z)$ such that $g : X \setminus E \to Z \setminus \{z\}$ is an isomorphism.*

Proof. Pick an $f$-very ample divisor $H$ on $X$ such that $R^1 f_* \mathcal{O}_X(nH) = 0$ for $n \geq 1$. Using the sequences

$$0 \to \mathcal{O}_X(nH + iE) \to \mathcal{O}_X(nH + (i+1)E) \to \mathcal{O}_X(nH + (i+1)E)|_E \to 0$$

we see that

(1) $|nH + iE|$ is very ample on $X \setminus E$ for $n, i \geq 0$,
(2) $R^1 f_* \mathcal{O}_X(nH + iE) = 0$ if $((nH + iE) \cdot E) \geq -1$,
(3) $f_* \mathcal{O}_X(nH+iE) \twoheadrightarrow H^0(E, \mathcal{O}_E((nH+iE)|_E))$ is onto if $((nH+iE) \cdot E) \geq 0$, and
(4) $|nH + iE|$ is base point free if $((nH + iE) \cdot E) \geq 0$.

By the adjunction formula, $\deg \omega_E = (E \cdot E) + (E \cdot K_X)$ is the sum of 2 negative numbers. Thus, by (100), $\mathcal{O}_X(E)|_E \cong \mathcal{O}_X(K_X)|_E$ are negative generators of $\operatorname{Pic} E$. In particular, for any line bundle $L$, $(L \cdot E)/(E \cdot E)$ is an integer.

Set $m = (H \cdot E)/(E \cdot E)$. Then $\big((H + mE) \cdot E\big) = 0$, $\big((H + (m-1)E) \cdot E\big) = -(E \cdot E)$ and by (3) there is a curve $D \in |H + (m-1)E|$ which does not contain $E$. Since $\mathcal{O}_X(D)|_E$ is a generator of $\operatorname{Pic} E$, we conclude that $D$ is regular at $D \cap E$. By (4) we can choose $D' \in |H + mE|$ disjoint from $E$. From the sequence

$$0 \to \mathcal{O}_X((n-1)(H + mE) + E) \to \mathcal{O}_X(n(H + mE)) \to \mathcal{O}_D(n(H + mE)|_D) \to 0$$

we conclude that $f_* \mathcal{O}_X(n(H + mE)) \twoheadrightarrow f_* \mathcal{O}_D(n(H + mE)|_D)$ is onto for $n \gg 1$. Moreover, $n(H + mE)|_D$ is very ample for $n \gg 1$.

Let $g : X \to Z$ be the morphism given by $|n(H + mE)|$ for some $n \gg 1$. As we saw,

(5) $g$ contracts $E$ to a point $z$,
(6) $|n(H + mE)|$ is very ample on $X \setminus E$, and
(7) $|n(H + mE)|$ is also very ample on $D$.

Thus $g(D + (n-1)D') = g(E + D + (n-1)D') \subset Z$ is a curve which is regular at $z$. It is also a hyperplane section of $Z$ since $E + D + (n-1)D' \in |n(H + mE)|$, thus $Z$ is regular at $z$. □

LEMMA 100. *Let $k$ be a field and $C$ a reduced and irreducible $k$-curve such that $H^1(C, \mathcal{O}_C) = 0$ and $\omega_C$ is locally free. Then*

(1) *$H^1(C, L) = 0$ for every line bundle $L$ such that $\deg L \geq 0$.*
(2) *$\operatorname{Pic}(C) \cong \mathbb{Z}[H]$ where $H$ denotes the positive generator.*
(3) *Furthermore,*
   (a) *either $\omega_C \cong H^{-1}$ and $C$ is isomorphic to a conic,*
   (b) *or $\omega_C \cong H^{-2}$ and $C$ is isomorphic to a line,*
   *both in the projective plane over $\operatorname{Spec}_k H^0(C, \mathcal{O}_C)$.*

Proof. Since $H^1(C, \mathcal{O}_C)$ is the tangent space to $\operatorname{Pic}(C)$ (see [**Mum66**, Lect.24]) $\operatorname{Pic}(C)$ is reduced, hence isomorphic to $\mathbb{Z}$. Let $H$ be the positive generator.

Set $K := H^0(C, \mathcal{O}_C)$ and let $r = \deg[K : k]$. Then $\chi(\mathcal{O}_C) = r$ and by Riemann-Roch, $\chi(C, H^m) = m \deg H + r$, hence $h^0(C, H^m) > 0$ for $m \geq 0$. From

$$0 \to \mathcal{O}_C \to H^m \to (\text{torsion sheaf}) \to 0$$

we obtain that $H^1(C, H^m) = 0$ for $m \geq 0$.

Since $H^1(C, \omega_C) \neq 0$, this implies that $\omega_C \cong H^{-n}$ for some $n \geq 1$ and $-r = \chi(C, \omega_C) = -n \deg H + r$.

Since $H^0(C, H)$ and $H^1(C, H)$ are both $K$-vector spaces, we see that $\deg H$ is a multiple of $r$, say $\deg H = dr$. Thus $2r = ndr$ and so $nd = 2$, hence $n \in \{1, 2\}$ as claimed.

As an easy case of Castelnuovo-Mumford regularity (see, for instance [**Laz04**, Sec.1.8]), we see that $H$ is very ample, hence $C$ is a line or a conic in the projective plane over $\operatorname{Spec}_k K$. $\square$

101 (Nodes). We say that a scheme $S$ has a *node* at a point $s \in S$ if its local ring $\mathcal{O}_{s,S}$ can be written as $R/(f)$ where $(R, m)$ is a regular local ring of dimension 2, $f \in m^2$ and $f$ is not a square in $m^2/m^3$.

101.1 *Claim.* Let $(A, n)$ be a 1-dimensional local ring with residue field $k$ and normalization $\bar{A}$. Let $\bar{n}$ be the intersection of the maximal ideals of $\bar{A}$. Assume that $(A, n)$ is a quotient of a regular local ring. Then $(A, n)$ is nodal iff $\dim_k(\bar{A}/\bar{n}) = 2$ and $\bar{n} \subset A$.

Proof. If $(A, n)$ is nodal then blowing up $n$ gives the normalization and these properties are clear.

Conversely, if $\dim_k(\bar{A}/\bar{n}) = 2$ then $\dim_k(\bar{n}^r/\bar{n}^{r+1}) = 2$ for every $r$. Let $x, y \in \bar{n}$ be a $k$-basis of $\bar{n}/\bar{n}^2$. Then $x^2, xy, y^2$ are $k$-linearly dependent in $\bar{n}^2/\bar{n}^3$. The relation is not a perfect square since

$$(\bar{n}/\bar{n}^2) \otimes_{\bar{A}/\bar{n}} (\bar{n}/\bar{n}^2) \to (\bar{n}^2/\bar{n}^3) \quad \text{is an isomorphism.}$$

Thus $x, y$ generate $n$ and and the resulting relation is not a square modulo $n^3$. $\square$

101.2 *Deformation of nodes.* Let $(B', n')$ be a complete DVR with maximal ideal $n'$ and $(R, m)$ a complete, local $B'$-algebra such that $R$ is flat over $B'$.

If $B'/n'$ is perfect, or, more generally, if $R/m$ is separably generated over $B'/n'$, then there is a complete DVR $(B', n') \subset (B, n) \subset (R, m)$ such that $R$ is a quotient of a formal power series ring $B[[x_1, \dots, x_r]]$ (cf. [**Mat86**, Sec.29]).

If $R/nR$ has a node at $m$, then $R$ has dimenson 2 and embedding dimension 3, hence we can write

$$R \cong B[[x, y]]/\big(G(x, y)\big)$$

for some $G \in B[[x, y]]$ that defines a node over $B/n$. That is, there is a quadratic form $q(x, y) := ax^2 + bxy + cy^2$ with $a, b, c \in B$ such that

$$\left(\tfrac{\partial q}{\partial x}, \tfrac{\partial q}{\partial y}\right) = (x, y) \quad \text{and} \quad G - q(x, y) \in nB[[x, y]] + (x, y)^3.$$

Assume now that there are coordinates $(x_r, y_r)$ such that $G - q(x_r, y_r) \in m^r(x, y) + n$. For suitable $h_x, h_y \in m^r$, we can define new coordinates $x_{r+1} := x_r + h_x, y_{r+1} := y_r + h_y$ such that $G - q(x_{r+1}, y_{r+1}) \in m^{r+1}(x, y) + n$. We reapeat this until, at the end, in suitable coordinates $(x_\infty, y_\infty)$,

$$R \cong B[[x_\infty, y_\infty]]/\big(ax_\infty^2 + bx_\infty y_\infty + cy_\infty^2 + d\big) \quad \text{where } a, b, c \in B \text{ and } d \in n.$$

If $\operatorname{char} B/n \neq 2$ then this can be further simplified to $B[[x, y]]/\big(x^2 + cy^2 + d\big)$ where $c \in B \setminus n$ and $d \in n$. If, in addition, $B/n$ is algebraically closed, then to $B[[x, y]]/\big(xy + d\big)$.

Finally, if $B$ is a power series ring $\mathbb{C}[[z]]$ then we obtain the simplest form

$$R \cong \mathbb{C}[[x, y, z]]/(xy + z^r) \quad \text{for some } r \geq 1.$$

## 4. Rational pairs – JK and SK

One of the early encouraging results about canonical singularities was the, quite subtle, proof by [**Elk81**] that they are rational (48).

The aim of this sections is two-fold. First, we give rather general results that connect CM sheaves and rationality on $X$ with various vanishing results on a resolution of $X'$ (111) and (113).

Then we develop a notion of rational pairs (117). This concept seems to capture the rationality properties of dlt pairs (120).

Note that everything before (119) works in arbitrary characteristic.

DEFINITION 102. Let $A$ be a local ring and $M$ an $A$-module. We say that $M$ is *Cohen-Macaulay* (or *CM* for short) if $\operatorname{depth} M = \dim M$, see [**Mat86**, Sec.17].

Let $X$ be a noetherian scheme. As in (48), a quasi coherent sheaf $\mathcal{F}$ on $X$ is called *Cohen-Macaulay* (or *CM*) if $\mathcal{F}_x$ is a Cohen-Macaulay module over $\mathcal{O}_{X,x}$ for all closed points $x \in \operatorname{Supp} \mathcal{F}$. Note that if $X$ is a closed subscheme of $Y$ then $\mathcal{F}$ is CM as an $\mathcal{O}_X$-sheaf iff it is CM as an $\mathcal{O}_Y$-sheaf.

By Grothendieck's vanishing theorem (see [**Gro67**, Sec.3] or [**BH93**, 3.5.7]) $\mathcal{F}$ is CM at $x$ if and only if

$$H^i_x(X, \mathcal{F}) = 0 \quad \text{for all } i \neq \dim \mathcal{F}_x.$$

$X$ is called CM if $\mathcal{O}_X$ is. Finally, $X$ is called *Gorenstein* if $\mathcal{O}_X$ is CM and its dualizing sheaf $\omega_X$ is an invertible sheaf.

LEMMA 103. *If $\mathcal{F}$ is CM at $x$, then* $\operatorname{Supp} \mathcal{F}$ *is equidimensional in a neighborhood of $x$. In particular, if $\mathcal{F}$ is CM and* $\operatorname{Supp} \mathcal{F}$ *is connected, then* $\operatorname{Supp} \mathcal{F}$ *is equidimensional.*

PROOF. Let $d_{\min}$ and $d_{\max}$ denote the smallest and largest dimension of the components of $\operatorname{Supp} \mathcal{F}$ containing $x$. Then by [**BH93**, 1.2.13],

$$\operatorname{depth} \mathcal{F}_x \leq d_{\min} \leq d_{\max} = \dim_x \operatorname{Supp} \mathcal{F} = \dim \mathcal{F}_x.$$

If $\mathcal{F}_x$ is CM, then the two sides of this inequality are equal, forcing that $d_{\min} = d_{\max}$. $\square$

COROLLARY 104. *Let $x \in X$ be a closed point and*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

*a sequence of coherent sheaves on $X$ which is exact at $x$. Assume that* $\operatorname{Supp} \mathcal{F}'$ *and* $\operatorname{Supp} \mathcal{F}$ *are equidimensional of dimension $d$ in a neighborhood of $x$.*

(1) *If $\mathcal{F}'$ and $\mathcal{F}''$ are CM at $x$, and $\dim \mathcal{F}''_x = d$, then $\mathcal{F}$ is also CM at $x$.*
(2) *If $\mathcal{F}$ and $\mathcal{F}''$ are CM at $x$, and $d-1 \leq \dim \mathcal{F}''_x \leq d$, then $\mathcal{F}'$ is also CM at $x$.*
(3) *If $\mathcal{F}'$ and $\mathcal{F}$ are CM and $\dim \mathcal{F}''_x < d$, then $\mathcal{F}''$ is also CM at $x$. It follows that in this case $\dim \mathcal{F}''_x = d-1$.*

PROOF. Follows from the long exact sequence of local cohomology at $x$ and (102). $\square$

**CM criteria and the $\omega$-dual.**

DEFINITION 105. [**Har66, AK70, Con00**] Let $X$ be a scheme. If $X$ admits a dualizing complex, it is denoted by $\omega_X^{\bullet}$. Note that if $X$ is of pure dimension $n$, the dualizing sheaf of $X$ is $\omega_X := h^{-n}(\omega_X^{\bullet})$. A relatively straightforward consequence of the definition of the dualizing sheaf and basic properties of CM rings is that $X$ is CM if and only if $\omega_X^{\bullet} \simeq_{qis} \omega_X[n]$.

ASSUMPTION 106. For the rest of this section we assume that every scheme admits a dualizing complex and can be locally embedded as a closed subscheme into a Gorenstein scheme. This holds, for instance, if $X$ is of finite type over a field.

DEFINITION 107. Let $\mathcal{F}$ be a coherent sheaf on $X$ with equidimensional support of dimension $d = \dim \mathcal{F}$. We define the *$\omega$-dual* of $\mathcal{F}$ to be the coherent sheaf

$$\mathbb{d}(\mathcal{F}) := \mathcal{E}xt_X^{-d}(\mathcal{F}, \omega_X^{\bullet}) = h^{-d}(R\mathcal{H}om_X(\mathcal{F}, \omega_X^{\bullet})).$$

Notice that if $X$ is CM and $\mathcal{F}$ is locally free, then this agrees with the usual dual of $\mathcal{F}$ twisted by the dualizing sheaf : $\mathbb{d}(\mathcal{F}) \simeq \mathcal{H}om(\mathcal{F}, \omega_X) \simeq \mathcal{F}^* \otimes \omega_X$. In fact, something similar holds in general as shown in the next lemma.

LEMMA 108. *Let $\mathcal{F}$ be a coherent sheaf on $X$ with equidimensional support of dimension $d = \dim \mathcal{F}$. Further let $Z \subseteq X$ be a subscheme that contains the support of $\mathcal{F}$. Then $\mathbb{d}(\mathcal{F}) \simeq \mathcal{E}xt_Z^{-d}(\mathcal{F}, \omega_Z^{\bullet})$. In particular, if $\dim Z = d$, then $\mathbb{d}(\mathcal{F}) \simeq \mathcal{H}om_Z(\mathcal{F}, \omega_Z)$.*

PROOF. Let $\iota : Z \to X$ denote the embedding of $Z$ into $X$. Then by Grothendieck duality and the definition of $\omega_Z$,

$$\begin{aligned} \mathbb{d}(\mathcal{F}) &\simeq \mathcal{E}xt_X^{-d}(\mathcal{F}, \omega_X^{\bullet}) \simeq h^{-d}(R\mathcal{H}om_X(\mathcal{F}, \omega_X^{\bullet})) \\ &\simeq h^{-d}(R\mathcal{H}om_Z(\mathcal{F}, \omega_Z^{\bullet})) \simeq \mathcal{E}xt_Z^{-d}(\mathcal{F}, \omega_Z^{\bullet}). \end{aligned}$$

If $\dim Z = d$, then $\mathcal{E}xt_Z^{-d}(\mathcal{F}, \omega_Z^{\bullet}) \simeq \mathcal{H}om_Z(\mathcal{F}, \omega_Z)$. Since $\mathcal{F}$ is coherent, this also implies that so is $\mathbb{d}(\mathcal{F})$. $\square$

THEOREM 109. *A sheaf $\mathcal{F}$ is CM at $x \in X$ if and only if*

$$\big(R\mathcal{H}om_X(\mathcal{F}, \omega_X^{\bullet})\big)_x \simeq_{qis} \mathbb{d}(\mathcal{F})_x[\,d\,].$$

PROOF. We may obviously assume that $\mathcal{F} \neq 0$. First we will show that $\mathcal{F}$ is CM at $x \in X$ if and only if

$$\big(\mathcal{E}xt_X^i(\mathcal{F}, \omega_X^{\bullet})\big)_x = 0$$

for all but a single value of $i$.

As the statement is local we may assume that $X$ itself is embedded into a Gorenstein scheme as a closed subscheme. Let $j : X \hookrightarrow Y$ be such an embedding and $x \in X$ a closed point. Let $N = \dim Y$. Using first that $j_*$ is exact, then Grothendieck duality and finally that $\omega_Y^{\bullet} \simeq_{qis} \omega_Y[N]$ we obtain that

$$\begin{aligned} R\mathcal{H}om_X(\mathcal{F}, \omega_X^{\bullet}) &\simeq_{qis} Rj_*R\mathcal{H}om_X(\mathcal{F}, \omega_X^{\bullet}) \\ &\simeq_{qis} R\mathcal{H}om_Y(Rj_*\mathcal{F}, \omega_Y^{\bullet}) \\ &\simeq_{qis} R\mathcal{H}om_Y(\mathcal{F}, \omega_Y)[N]. \end{aligned}$$

Then taking cohomology and localizing at $x$ gives that

$$
\begin{aligned}
\big(\mathcal{E}xt_X^{-i}(\mathcal{F},\omega_X^{\bullet})\big)_x &\simeq \big(h^{-i}(R\mathcal{H}om_X(\mathcal{F},\omega_X^{\bullet}))\big)_x \simeq \big(h^{-i}(Rj_*R\mathcal{H}om_X(\mathcal{F},\omega_X^{\bullet}))\big)_x\\
&\simeq \big(h^{-i}(R\mathcal{H}om_Y(\mathcal{F},\omega_Y^{\bullet}))\big)_x \simeq \big(h^{-i}(R\mathcal{H}om_Y(\mathcal{F},\omega_Y)[N])\big)_x\\
&\simeq \big(\mathcal{E}xt_Y^{N-i}(\mathcal{F},\omega_Y)\big)_x \simeq \operatorname{Ext}^{N-i}_{\mathcal{O}_{Y,x}}(\mathcal{F}_x,\omega_{\mathcal{O}_{Y,x}}).
\end{aligned}
$$

Observe, that the right hand side term in this isomorphism is the dual of $H^i_x(Y,\mathcal{F})$ by local duality [**BH93**, 3.5.9]. Now as $X$ is closed in $Y$, $\mathcal{F}$ is CM at $x$ over $X$ if and only if it is CM at $x$ over $Y$, so by (102) $\mathcal{F}$ is CM at $x$ (either over $X$ or over $Y$) if and only if $\operatorname{Ext}^{N-i}_{\mathcal{O}_{Y,x}}(\mathcal{F}_x,\omega_{\mathcal{O}_{Y,x}})=0$ for all but a single value of $i$. This proves the desired statement.

Denote by $d$ the value of $i$ for which the above group is non-zero. Since [**BH93**, 3.5.11] implies that $\operatorname{Ext}^{N-\dim\mathcal{F}_x}_{\mathcal{O}_{Y,x}}(\mathcal{F}_x,\omega_{\mathcal{O}_{Y,x}})\neq 0$, we obtain that $d=\dim\mathcal{F}_x$. By the definition of $\mathscr{d}(\mathcal{F})$ this completes the proof of the theorem. □

Corollary 110. *Let $\mathcal{F}$ be a CM sheaf. Then $\mathscr{d}(\mathcal{F})$ is also CM, and*

$$R\mathcal{H}om_X(\mathcal{F},\omega_X^{\bullet})\simeq_{qis}\mathscr{d}(\mathcal{F})[d].$$

*Furthermore,* $\operatorname{Supp}\mathscr{d}(\mathcal{F})=\operatorname{Supp}\mathcal{F}$ *and* $\mathscr{d}(\mathscr{d}(\mathcal{F}))\simeq\mathcal{F}$.

A special case of this was proved in [**KM98**, 5.70]. The fact that it is indeed a special case follows from (108).

Proof. As a direct consequence of (109) we get that $R\mathcal{H}om_X(\mathcal{F},\omega_X^{\bullet})\simeq_{qis}\mathscr{d}(\mathcal{F})[d]$. Then, since $\omega_X^{\bullet}$ is the dualizing complex,

$$R\mathcal{H}om(\mathscr{d}(\mathcal{F}),\omega_X^{\bullet})\simeq_{qis}R\mathcal{H}om(R\mathcal{H}om(\mathcal{F},\omega_X^{\bullet})[-d],\omega_X^{\bullet})\simeq_{qis}\mathcal{F}[d].$$

This in turn implies that $\operatorname{Supp}\mathscr{d}(\mathcal{F})=\operatorname{Supp}\mathcal{F}$ and $\mathscr{d}(\mathscr{d}(\mathcal{F}))\simeq\mathcal{F}$, and, by (109 again, that $\mathscr{d}(\mathcal{F})$ is CM. □

Theorem 111. *Let $f:\widetilde{X}\to X$ be a proper morphism and $\mathcal{G}$ a CM sheaf with equidimensional support of dimension $d=\dim\mathcal{G}$ on $\widetilde{X}$. Assume that there exist two integers $a,b\in\mathbb{Z}$ such that $R^if_*\mathcal{G}=0$ for $i\neq a$ and $R^jf_*(\mathscr{d}(\mathcal{G}))=0$ for $j\neq b$. Then $\mathcal{F}:=R^af_*\mathcal{G}$ is a CM sheaf on $X$ of dimension $d+a-b$ with $\mathscr{d}(\mathcal{F})\simeq R^bf_*(\mathscr{d}(\mathcal{G}))$.*

Proof. By assumption,

$$
\begin{aligned}
&R\mathcal{H}om_X(\mathcal{F},\omega_X^{\bullet})\simeq_{qis}R\mathcal{H}om_X(Rf_*\mathcal{G}[-a],\omega_X^{\bullet})\simeq\\
&\qquad Rf_*R\mathcal{H}om_{\widetilde{X}}(\mathcal{G},\omega_{\widetilde{X}}^{\bullet})[a]\simeq Rf_*(\mathscr{d}(\mathcal{G}))[d+a]\simeq R^bf_*(\mathscr{d}(\mathcal{G}))[d+a-b].
\end{aligned}
$$

Then the statement follows from Theorem 109. □

Corollary 112. *Let $f:\widetilde{X}\to X$ be a proper morphism and $\mathcal{G}$ a CM sheaf with equidimensional support of dimension $d=\dim\mathcal{G}$ on $\widetilde{X}$ such that $R^if_*\mathcal{G}=0$ and $R^if_*(\mathscr{d}(\mathcal{G}))=0$ for $i>0$. Then $\mathcal{F}:=f_*\mathcal{G}$ is a CM sheaf on $X$ of dimension $d$ with $\mathscr{d}(\mathcal{F})\simeq f_*(\mathscr{d}(\mathcal{G}))$.* □

**Vanishing theorems.**

The following is a generalization of [**Kov00**, Thm.1].

THEOREM 113. *Let $f : Y \to X$ be a proper birational morphism of pure dimensional schemes and $\mathcal{G}$ a CM sheaf on $Y$. Assume that*

(1) $\operatorname{Supp}\mathcal{G} = Y$,
(2) $R^i f_* \mathscr{d}(\mathcal{G}) = 0$ *for* $i > 0$, *and*
(3) *The natural map* $\rho : f_*\mathcal{G} \to Rf_*\mathcal{G}$ *admits a left-inverse* $\rho' : Rf_*\mathcal{G} \to f_*\mathcal{G}$, *that is,* $\rho' \circ \rho$ *is the identity of* $f_*\mathcal{G}$.

*Then $f_*\mathcal{G}$ is a CM sheaf on $X$ and $R^i f_*\mathcal{G} = 0$ for $i > 0$.*

(Strictly speaking the automorphism $\rho' \circ \rho$ lives in the derived category of coherent $\mathcal{O}_X$-modules, and so it is an auto-quasi-isomorphism, but since $f_*\mathcal{G}$ is a sheaf, $h^0$ of the derived category auto-quasi-isomorphism induces an honest sheaf automorphism of $f_*\mathcal{G}$.)

PROOF. Consider the morphisms $\rho$ and $\rho'$:

$$\begin{tikzcd} f_*\mathcal{G} \arrow[r, "\rho"] \arrow[rr, bend right, "\simeq"'] & Rf_*\mathcal{G} \arrow[r, "\rho'"] & f_*\mathcal{G}, \end{tikzcd}$$

and apply the functor $R\mathcal{H}om(\_, \omega_X^{\bullet})$;

$$\begin{tikzcd} R\mathcal{H}om(f_*\mathcal{G}, \omega_X^{\bullet}) \arrow[r] \arrow[rr, bend right, "\simeq"'] & R\mathcal{H}om(Rf_*\mathcal{G}, \omega_X^{\bullet}) \arrow[r] & R\mathcal{H}om(f_*\mathcal{G}, \omega_X^{\bullet}). \end{tikzcd}$$

Applying Grothendieck duality, (110) and (113.2) to the middle term yields that

$$R\mathcal{H}om_X(Rf_*\mathcal{G}, \omega_X^{\bullet}) \simeq_{qis} Rf_*R\mathcal{H}om_Y(\mathcal{G}, \omega_Y^{\bullet}) \simeq_{qis} Rf_*\mathscr{d}(\mathcal{G})[n] \simeq_{qis} f_*\mathscr{d}(\mathcal{G})[n]. \tag{113.4}$$

This implies that the automorphism of $h^i\big(R\mathcal{H}om(f_*\mathcal{G}, \omega_X^{\bullet})\big)$ induced by (113.4) factors through 0 for $i \neq d$. Therefore

$$R\mathcal{H}om_X(f_*\mathcal{G}, \omega_X^{\bullet}) \simeq_{qis} \mathcal{H}[n]$$

for some sheaf $\mathcal{H}$ and we obtain that the induced automorphism of $\mathcal{H}$ factors through $f_*(\mathscr{d}(\mathcal{G}))$:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\alpha"] \arrow[rr, bend right, "\simeq"'] & f_*(\mathscr{d}(\mathcal{G})) \arrow[r, "\beta"] & \mathcal{H}. \end{tikzcd}$$

Since $f_*(\mathscr{d}(\mathcal{G}))$ is torsion-free and both $\alpha$ and $\beta$ are generically isomorphisms, it follows that they are isomorphisms everywhere. In other words, we conclude that

$$R\mathcal{H}om_X(f_*\mathcal{G}, \omega_X^{\bullet}) \simeq_{qis} \mathcal{H}[n] \simeq f_*\mathscr{d}(\mathcal{G})[n] \simeq_{qis} Rf_*\mathscr{d}(\mathcal{G})[n].$$

Finally consider the following sequence of isomorphisms:

$$f_*\mathcal{G} \simeq_{qis} R\mathcal{H}om_X\left(R\mathcal{H}om_X(f_*\mathcal{G}, \omega_X^{\bullet}), \omega_X^{\bullet}\right) \simeq_{qis} R\mathcal{H}om_X\left(Rf_*\mathscr{d}(\mathcal{G})[n], \omega_X^{\bullet}\right) \simeq_{qis}$$
$$\simeq_{qis} Rf_*R\mathcal{H}om_Y\left(\mathscr{d}(\mathcal{G})[n], \omega_Y^{\bullet}\right) \simeq_{qis} Rf_*R\mathcal{H}om_Y\left(R\mathcal{H}om_Y(\mathcal{G}, \omega_Y^{\bullet}), \omega_Y^{\bullet}\right) \simeq_{qis} Rf_*\mathcal{G}.$$

It follows that $R^i f_*\mathcal{G} = 0$ for $i > 0$. Finally, the statement that $f_*\mathcal{G}$ is CM follows from (112). □

We will only use the following special case of (113) in the sequel.

COROLLARY 114. *Let $f : Y \to X$ be a proper birational morphism and $\mathcal{L}$ an invertible sheaf on $Y$. Assume that*

1. *$Y$ is Gorenstein of dimension $n$,*
2. *$R^i f_*\big(\omega_Y \otimes \mathcal{L}\big) = 0$ for $i > 0$, and*
3. *The natural map $\rho : f_*\mathcal{L}^{-1} \to Rf_*\mathcal{L}^{-1}$ admits a left-inverse $\rho' : Rf_*\mathcal{L}^{-1} \to f_*\mathcal{L}^{-1}$, that is, $\rho' \circ \rho$ is the identity of $f_*\mathcal{L}^{-1}$.*

*Then $R^i f_*\mathcal{L}^{-1} = 0$ for $i > 0$.* □

It is through the condition (114.2) that the characteristic 0 assumption enters into many of the applications. For the first 2 cases of the next vanishing theorem see [**KM98**, 2.64]. The last case follows from this by a simple induction. A more general version is in [**Fuj08**, ??].

THEOREM 115. *Let $Y$ be a smooth variety over a field of characteristic 0, $f : Y \to X$ a proper morphism and $L$ a $\mathbb{Z}$-divisor on $Y$. Assume that $L \sim_{\mathbb{Q},f} M + \Delta$ where $M$ is an $f$-nef $\mathbb{Q}$-divisor and $\Delta = \sum a_i D_i$ an snc divisor with $0 \leq a_i \leq 1$ for every $i$. Assume that one of the following holds:*

1. *$a_i < 1$ for every $i$ and $f$ is birational,*
2. *$a_i < 1$ for every $i$ and $M$ is $f$-big,*
3. *$a_i \leq 1$ for every $i$ and $M$ is $f$-big on every lc center (130) of $(X, \Delta)$.*

*Then $R^i f_*\big(\omega_Y(L)\big) = 0$ for every $i > 0$.* □

**Rational pairs.**

DEFINITION 116. Let $(X, D)$ be a pair such that $D$ is an integral divisor. We say that $(X, D)$ is a *normal pair (resp. seminormal pair) if there exists a log resolution (resp. log semi-resolution (**??**?)) $f : (Y, D_Y) \to (X, D)$ such that the natural map $\mathcal{O}_X(-D) \to f_*\mathcal{O}_Y(-D_Y)$ is an isomorphism.*

DEFINITION 117. Let $(X, D)$ be a pair such that $D$ is a reduced integral divisor. Then $(X, D)$ is said to be a *rational pair* (resp. *semirational pair*) if there exists a log resolution (resp. log semi-resolution) $f : (Y, D_Y) \to (X, D)$ such that

1. $\mathcal{O}_X(-D) \simeq f_*\mathcal{O}_Y(-D_Y)$,
2. $R^i f_*\mathcal{O}_Y(-D_Y) = 0$ for $i > 0$, and
3. $R^i f_*\omega_Y(D_Y) = 0$ for $i > 0$.

We check in (119) that these properties hold for every log resolution (resp. log semi-resolution).

Note that $(X, \emptyset)$ is a rational pair if and only if $X$ has rational singularities according to the usual definition [**KM98**, 5.8].

Note that [**ST08**] defines a notion of rational singularities of pairs $(X, D)$ such that $\lfloor D \rfloor = 0$. We are, however, particularly interested in the case when $D$ is an integral divisor. Investigating the relationship between the two definitions is left to the interested reader.

LEMMA 118. *Using the notation of* (117) *the conditions* (117.1-3) *are equivalent to the following:*

1. *$\mathcal{O}_X(-D) \simeq_{qis} Rf_*\mathcal{O}_Y(-D_Y)$, and*
2. *$\omega_X(D) \simeq_{qis} R\mathcal{H}om_X(\mathcal{O}_X(-D), \omega_X^{\bullet})[n]$.*

PROOF. Easily, (117.1-2) are equivalent to (118.1). Let $n = \dim X$. Then Grothendieck duality yields that

$$\begin{aligned} R\mathcal{H}om_X(\mathcal{O}_X(-D), \omega_X^{\bullet}) &\simeq_{qis} R\mathcal{H}om_X(Rf_*\mathcal{O}_Y(-D_Y), \omega_X^{\bullet}) \\ &\simeq_{qis} Rf_*R\mathcal{H}om_Y(\mathcal{O}_Y(-D_Y), \omega_Y^{\bullet}) \simeq_{qis} Rf_*\omega_Y(D_Y). \end{aligned}$$

Thus $R^i f_*\omega_Y(D_Y) = 0$ for $i > 0$ iff $R\mathcal{H}om_X(\mathcal{O}_X(-D), \omega_X^{\bullet})$ is quasi-isomorphic to its degree $-n$ cohomology, which is $\mathcal{H}om_X(\mathcal{O}_X(-D), \omega_X) \cong \omega_X(D)$. □

LEMMA 119. *Let $X$ be a scheme of finite type over a field of characteristic 0. Then the definition of a pair being rational (resp. semirational) as in (117) is independent of the log resolution chosen.*

PROOF. Let us check this first in case $(Y, B)$ is an snc pair and $g : (Z, A) \to (Y, B)$ a log resolution. Note that (118.2) is clear since $\omega_X^{\bullet} = \omega_X[-n]$.

We check (118.1) using induction on $\dim X$. Let $B = B_1 + B_+$ where $B_1$ is irreducible and denote the corresponding decomposition on $Z$ by $A = A_1 + A_+$. Notice that then $(A_1, A_+|_{A_1}) \to (B_1, B_+|_{B_1})$ is a log resolution. Then considering the short exact sequence,

$$0 \to \mathcal{O}_Z(-A_+) \to \mathcal{O}_Z(-A) \to \mathcal{O}_{B_1}(-A|_{B_1}) \to 0,$$

shows that snc pairs are rational by induction on the dimension of $Y$ and the number of components of $B$. Now if $B' \leq B$ is any effective divisor, then $(Y, B')$ is again snc, hence rational. Therefore the desired statement follows.

Next let $(X, D)$ be any pair and $f_i : (Y_i, D_i) \to (X, D)$ two log resolutions. Then there exists an snc pair $(Z, A)$ and common log resolutions $g_i : (Z, A) \to (Y_i, D_i)$ such that $f_1 \circ g_1 = f_2 \circ g_2$. Notice that $(g_1)_*^{-1} D_1 = (g_2)_*^{-1} D_2$, so one can use the same $A$ for both $g_1$ and $g_2$. Therefore

$$\begin{aligned} R(f_1)_*\mathcal{O}_{Y_1}(-D_1) &\simeq_{qis} R(f_1)_*R(g_1)_*\mathcal{O}_Z(-A) \\ &\simeq_{qis} R(f_2)_*R(g_2)_*\mathcal{O}_Z(-A) \simeq_{qis} R(f_2)_*\mathcal{O}_{Y_2}(-D_2). \end{aligned}$$

By (118) this proves the desired statement. □

One of the main applications is the following, see [**KM98**, 5.25] and [**Fuj08**, 4.14].

THEOREM 120. *Let $X$ be a scheme of finite type over a field of characteristic 0, $D$ a $\mathbb{Z}$-divisor and $L$ a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor. Assume that $(X, \Delta)$ is dlt for some effective $\mathbb{Q}$-divisor $\Delta$ and $D \leq \lfloor \Delta \rfloor$. Then*

(1) *$\mathcal{O}_X$ is CM,*
(2) *$\mathcal{O}_X(-D - L)$ is CM,*
(3) *$\omega_X(D + L)$ is CM,*
(4) *if $D + L$ is effective then $\mathcal{O}_{D+L}$ is CM, and*
(5) *$(X, D)$ is rational.*

PROOF. All the statements are local, so we may assume that $X$ is quasi-projective. Observe that (120.2) implies (120.1) by choosing $D = L = 0$ and (120.3) by (110). (120.1) and (120.2) imply (120.4) using (104) for the exact sequence

$$0 \to \mathcal{O}_X(-D - L)) \to \mathcal{O}_X \to \mathcal{O}_{D+L} \to 0.$$

We prove (120.5) by checking the conditions of (114) with $\mathcal{L} = \mathcal{O}_Y(-D_Y)$. By [**KM98**, 2.43] there exists an effective $\mathbb{Q}$-divisor $\Delta'$ such that $\lfloor \Delta' \rfloor = 0$ and $(X, D+$

$\Delta'$) has dlt singularities. Replacing $\Delta$ with $D+\Delta'$ we may assume that $\lfloor \Delta - D \rfloor = 0$.

By (**??**) there is a log resolution $f : (Y, \Delta_Y) \to (X, \Delta)$ such that every exceptional divisor has discrepancy $> -1$. That is, every exceptional divisor appears in $\Delta_Y$ with coefficient $< 1$. Thus we can write $\Delta_Y = D_Y + A - B$ where $A$ is exceptional with $\lfloor A \rfloor = 0$ and $B$ is exceptional and integral. Thus

$$B - D_Y \sim_{\mathbb{Q}} K_Y - f^*(K_X + \Delta) + A.$$

Since $f^*(K_X+\Delta)$ is numerically $f$-trivial and $\lfloor A \rfloor = 0$, $R^i f_* \mathcal{O}_Y(B - D_Y) = 0$ for $i > 0$ by [**KM98**, 2.68].

Now $B$ is an effective exceptional divisor, so we obtain that the composition,

$$\begin{tikzcd} \mathcal{O}_X(-D) \arrow[r] \arrow[rrr, bend right, "\simeq_{qis}"'] & Rf_*\mathcal{O}_Y(-D_Y) \arrow[r] & Rf_*\mathcal{O}_Y(B - D_Y) \arrow[r, "\simeq_{qis}"] & f_*\mathcal{O}_Y(B - D_Y) \end{tikzcd}$$

is a quasi-isomorphism. This shows that (114.3) holds.

By construction, $\mathcal{O}_Y$ is $f$-big on all log canonical centers of $(Y, D_Y)$. It is also obviously $f$-nef and $f$-big, hence we obtain by (115) that $R^i f_* \omega_Y(D_Y) = 0$ for $i > 0$, which is (114.2). Thus (114) implies that $R^i f_* \mathcal{O}_Y(-D_Y) = 0$ for $i > 0$, proving (120.5).

Next let $f : (Y, D_Y) \to (X, D)$ be a log resolution and $\mathcal{G} := \mathcal{O}_Y(-D_Y)$. Then $d(\mathcal{G}) = \omega_Y(D_Y)$ and so the $L = 0$ case of (120.2) follows from (112) and the definition of rationality (117).

Finally, we reduce (120.2) to the case when $L = 0$. Let us start with the additional assumption that $mL \sim 0$ for some $m > 0$. Let $\pi : X' \to X$ be the corresponding cyclic cover. Then $(X', \pi^*(D))$ is dlt (cf. [**KM98**, 5.20]) and $\mathcal{O}_X(-D-L)$ is a direct summand of $\pi_* \mathcal{O}_{X'}(-\pi^* D)$. Thus $\mathcal{O}_X(-D-L)$ is CM if $\mathcal{O}_{X'}(-\pi^* D)$ is by [**KM98**, 5.4] (cf. (112)). Now observe that the assumption $mL \sim 0$ always holds locally on $X$ and being CM is a local property, so we may assume that $L = 0$. $\square$

## 5. Adjunction

Adjunction is a classical method that relates the canonical class of a variety and the canonical class of a divisor. It is a very useful tool that allows lifting information from divisors to the ambient variety and facilitates induction on the dimension.

Definition 121 (Poincaré residue map I.). Let $X$ be a smooth variety over a field $k$ and $S \subset X$ a smooth divisor. Let

$$\mathcal{R}_S : \omega_X(S) \to \omega_S$$

(or $\mathcal{R}_{X \to S}$ if the choice of $X$ is not clear) denote the *Poincaré residue map*. It can be defined in two equivalent ways.

(Local definition.) At a point $s \in S \subset X$ choose local coordinates $x_1, \dots, x_n$ such that $S = (x_1 = 0)$. Then

$$\frac{dx_1}{x_1} \wedge dx_2 \wedge \dots \wedge dx_n$$

is a local generator of $\omega_X(S) = \Omega^n_X(S)$. Set

$$\mathcal{R}_S\Big(f \cdot \frac{dx_1}{x_1} \wedge dx_2 \wedge \dots \wedge dx_n\Big) = f|_S \cdot dx_2 \wedge \dots \wedge dx_n. \tag{121.1}$$

It is easy to check that $\mathcal{R}_S$ is independent of the local coordinates.

(Global definition.) View $\omega_X$ and $\omega_S$ as dualizing sheaves as in [**Har77**, III.7]. Then

$$\omega_S = \mathcal{E}xt^1_X(\mathcal{O}_S, \omega_X).$$

By applying $\mathcal{H}om(\ ,\omega_X)$ to the exact sequence

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

we get a long exact sequence

$$\mathcal{H}om_X(\mathcal{O}_X, \omega_X) \to \mathcal{H}om_X(\mathcal{O}_X(-S), \omega_X) \to \mathcal{E}xt^1_X(\mathcal{O}_S, \omega_X) \to \mathcal{E}xt^1_X(\mathcal{O}_X, \omega_X)$$

and the last term is zero since $\mathcal{O}_X$ is locally free. Thus we get the usual short exact sequence

$$0 \to \omega_X \to \omega_X(S) \stackrel{\mathcal{R}_S}{\to} \omega_S \to 0. \tag{121.2}$$

Note that this sequence is exact whenever $X$ is CM and $S \subset X$ has pure codimension 1. In particular, if $\omega_X(S)$ is locally free near $S$ then

$$\omega_X(S)|_S = \omega_S. \tag{121.3}$$

(Note that $\omega_S$ can be locally free even if $\omega_X(S)$ is not locally free. As an example, take $X = (xy - zt = 0) \subset \mathbb{A}^4$ and $S = (x = z = 0)$.) In many of our applications $K_X + S$ and $K_S$ are not Cartier but $m(K_X + S)$ and $mK_S$ are Cartier for some $m > 0$. By taking tensor powers, we get maps

$$\mathcal{R}_S^{\otimes m} : \bigl(\omega_X(S)\bigr)^{\otimes m} \to \omega_S^{\otimes m},$$

but we really would like to get a corresponding map between locally free sheaves

$$\omega_X^{[m]}(mS) := \Bigl(\bigl(\omega_X(S)\bigr)^{\otimes m}\Bigr)^{**} \stackrel{?}{\dashrightarrow} \Bigl(\omega_S^{\otimes m}\Bigr)^{**} =: \omega_S^{[m]}. \tag{121.4}$$

As we saw in (87), no such map exists in general, not even if $X$ is a normal surface. One needs a correction term, called the different (87.3).

Our next aim is to extend the definition of the different (and of the Poincaré residue map) to all dimensions. This does not seem always possible and in (122) I tried to list the minimal set of assumptions. They are somewhat numerous, but rather mild and satisfied in two important special cases:

- if $X$ is normal and $S' \to S$ is the normalization of a divisor,
- if $(X, S + \Delta)$ is lc or slc and $S' = S$.

Definition 122 (Different II.). Consider schemes and divisors over a perfect field $k$ satisfying the following conditions.

(1) $X$ is a reduced, pure dimensional scheme.
(2) $S \subset X$ is a reduced subscheme of pure codimension 1 and $X$ is smooth at all generic points of $S$.
(3) $D$ is a $\mathbb{Q}$-divisor on $X$ such that $X$ is smooth at all generic points of $D$ and no irreducible component of $S$ is contained in $\operatorname{Supp} D$.
(4) For some $m > 0$, the rank 1 reflexive sheaf $\omega_X^{[m]}(mS + mD)$ is locally free at all codimension 1 points of $S$.
(5) $S'$ is a reduced, pure dimensional scheme such that $\omega_{S'}$ is locally free in codimension 1.
(6) $\pi : S' \to S$ is a finite birational morphism (that is, finite and birational over each irreducible component).

By assumption, there is a closed subscheme $Z \subset X$ of codimension 2 such that $S$ and $X$ are both smooth at every point of $S \setminus Z$, $\pi : (S' \setminus \pi^{-1}Z) \to (S \setminus Z)$ is an isomorphism, $\omega_{S'}$ is locally free on $S' \setminus \pi^{-1}Z$ and $\operatorname{Supp} D \cap S \subset Z$.

Thus the Poincaré residue map (121) gives an isomorphism

$$\mathcal{R}_S : \omega_X(S)|_{(S\setminus Z)} \cong \omega_{(S\setminus Z)}.$$

Taking $m$th power gives an isomorphism

$$\mathcal{R}_S^m : \pi^*\omega_X^{[m]}(mS + mD)|_{(S'\setminus\pi^{-1}Z)} \cong \omega_{S'}^{[m]}|_{(S'\setminus\pi^{-1}Z)}.$$

By assumption, both $\omega_X^{[m]}(mS+mD)|_{S'}$ and $\omega_{S'}^{[m]}$ are locally free in codimension 1 (on $S'$). Therefore

$$\mathcal{H}om_{S'}\Big(\pi^*\omega_X^{[m]}(mS+mD), \omega_{S'}^{[m]}\Big)$$

is a rank 1 reflexive sheaf, locally free in codimension 1 and $\mathcal{R}_S^m$ defines a rational section. Hence there is a (not necessarily effective) divisor $D_{S'}$ which is Cartier in codimension 1 such that

$$\mathcal{R}_{S'}^m : \pi^*\omega_X^{[m]}(mS + mD) \cong \omega_{S'}^{[m]}\big(D_{S'}\big). \tag{122.7}$$

We formally divide by $m$ and define the *different* of $D$ on $S'$ as the $\mathbb{Q}$-divisor

$$\operatorname{Diff}_{S'}(D) := \tfrac{1}{m} D_{S'}. \tag{122.8}$$

We write the formula (122.7) in terms of $\mathbb{Q}$-divisors as

$$(K_X + S + D)|_{S'} \sim_{\mathbb{Q}} K_{S'} + \operatorname{Diff}_{S'}(D). \tag{122.9}$$

As in (7), the formula (122.9) has the problem that it indicates only that the two sides are $\mathbb{Q}$-linearly equivalent, whereas (122.7) is a *canonical* isomorphism.

Note that if $K_X + S$ and $D$ are both $\mathbb{Q}$-Cartier, then $\operatorname{Diff}_{S'}(0)$ is defined and

$$\operatorname{Diff}_{S'}(D) = \operatorname{Diff}_{S'}(0) + D|_S, \tag{122.10}$$

but in general the individual terms on the right do not make sense. If $S$ itself is Cartier, then (121.3) implies that $\operatorname{Diff}_{S'}(0) = 0$.

PROPOSITION 123. *Assume that $(X, S + D)$ is lc and $S' \to S$ is either the identity or the normalization $\pi : \bar{S} \to S$. Then:*

(1) *The different $\operatorname{Diff}_{S'}(D)$ is defined and it is an effective divisor.*
(2) $\operatorname{Diff}_{\bar{S}}(D) + K_{\bar{S}/S} = \pi^* \operatorname{Diff}_S(D)$.

Proof. Note that the different involves only divisors on $X$ and on $S'$, hence its computation involves only points of codimension 2 on $X$. Therefore we can localize at the generic point of a divisor $E \subset S$. Thus we may assume that $\dim X = 2$. For lc surfaces the different was computed in (87.3–5) and it is effective.

By (72), $S$ is either smooth or has a node at the generic point of $E$ and $E \not\subset \operatorname{Supp} D$. In the first case $\pi$ is an isomorphism near $E$ and (2) is clear. If $S$ has a node at the generic point of $E$ then $\operatorname{Diff}_S(D) = 0$ and $\operatorname{Diff}_{\bar{S}}(D)$ is the sum of the 2 preimages of the node (87.5). Thus $\operatorname{Diff}_{\bar{S}}(D) = -K_{\bar{S}/S}$, proving (2). $\square$

124 (Different and discrepancies). Let $(X, S+D)$ be a pair. Let $f : Y \to X$ be any proper birational morphism and $S_Y := f_*^{-1}S$. As in (7), write $K_Y + S_Y + D_Y \sim_{\mathbb{Q}} f^*(K_X + S + D)$ and assume that $\operatorname{Diff}_S(D)$ and $\operatorname{Diff}_{S_Y}(D_Y)$ are both defined. By

(122.9), $\big(K_Y + S_Y + D_Y\big)|_{S_Y} \sim_{\mathbb{Q}} K_{S_Y} + \operatorname{Diff}_{S_Y}(D_Y)$ and $(K_X + S + D)|_S \sim_{\mathbb{Q}} K_S + \operatorname{Diff}_S(D)$. Thus

$$K_{S_Y} + \operatorname{Diff}_{S_Y}(D_Y) \sim_{\mathbb{Q}} f^*\big(K_S + \operatorname{Diff}_S(D)\big). \tag{124.1}$$

Note that by (123),

$$K_{\bar S} + \operatorname{Diff}_{\bar S}(D) = \pi^*\big(K_S + \operatorname{Diff}_S(D)\big),$$

thus it matters very little whether we work with $S$ or $\bar S$. It is somewhat easier to use $\bar S$.

If $S_Y$ is smooth (or normal) then $f|_{S_Y} : S_Y \to S$ factors through the normalization as $f_S : S_Y \to \bar S$, and by (87.5) $\operatorname{Diff}_{\bar S}(D) = (f_S)_*\big(\operatorname{Diff}_{S_Y}(D_Y)\big)$. In particular, if $f : Y \to X$ is a log resolution, then $\operatorname{Diff}_{S_Y}(D_Y) = D_Y|_{S_Y}$ and we have the simpler formula

$$K_{S_Y} + D_Y|_{S_Y} \sim_{\mathbb{Q}} f_S^*\big(K_{\bar S} + \operatorname{Diff}_{\bar S}(D)\big) \quad \text{and} \quad \operatorname{Diff}_{\bar S}(D) = (f_S)_*\big(D_Y|_{S_Y}\big). \tag{124.2}$$

In order to get additional information, note that by further blowing up we may assume that $f_*^{-1}D$ is disjoint from $S_Y$ and if $E_i$ is an exceptional divisor of $f$ that intersects $S_Y$ then $\operatorname{center}_X E_i \subset S$. Note, however, that if $E_i$ is $f$-exceptional, $E_i \cap S_Y$ need not be $f_S$-exceptional. For such a resolution, (7) and (124.2) give the following.

*Claim 124.3.* Let $f : Y \to (X, S + D)$ be a (semi)resolution as above and $S_Y := f_*^{-1}S$ the birational transform of $S$.

a) Let $E \subset Y$ be an exceptional divisor and $E_D$ any irreducible component of $E \cap S_Y$. Then $a\big(E_D, \bar S, \operatorname{Diff}_{\bar S}(D)\big) = a(E, X, S + D)$.
b) Let $F \subset S_Y$ be an exceptional divisor and $F_X$ the divisor obtained by blowing up $F \subset Y$. Then $a\big(F, \bar S, \operatorname{Diff}_{\bar S}(D)\big) = a(F_X, X, S + D)$ and $\operatorname{center}_S F = \operatorname{center}_X F_X$.
c) Using the notation in (14),

$$\begin{aligned}&\operatorname{totaldiscrep}\big(\bar S, \operatorname{Diff}_{\bar S}(D)\big) = \\ &\qquad = \min_i\big\{a(E_i, X, S + D) : E_i \text{ is } f\text{-exceptional and } E_i \cap S_Y \neq \emptyset\big\}. \quad \square\end{aligned}$$

The condition $E_i \cap S_Y \neq \emptyset$ in (124.3.c) is rather difficult to control since it is not birational invariant. In order to get something that is visibly birational in nature, we consider the following variants of the discrepancy (14).

*Definition 124.4.* Let $(X, \Delta)$ be a pair and $Z \subset X$ a closed subscheme. Define

$$\begin{aligned}\operatorname{discrep}(\operatorname{center} \subset Z, X, \Delta) \quad &:= \quad \inf_E\{a(E, X, \Delta) : \operatorname{center}_X E \subset Z\}, \quad \text{and}\\ \operatorname{discrep}(\operatorname{center} \cap Z \neq \emptyset, X, \Delta) \quad &:= \quad \inf_E\{a(E, X, \Delta) : \operatorname{center}_X E \cap Z \neq \emptyset\},\end{aligned}$$

where, in both cases, $E$ runs through the set of all exceptional divisors over $X$. Both of these have a natural totaldiscrep version.

Finally, if $\mathcal{E}$ is a set of divisors over $X$ then inserting the conditions $E \notin \mathcal{E}$ means that we take the infimum over all divisors that are not in $\mathcal{E}$. For instance, if $[S]$ denotes the set of divisors corresponding to the irreducible components of some $S \subset X$, then

$$\operatorname{totaldiscrep}(E \notin [S], \operatorname{center} \cap S \neq \emptyset, X, \Delta)$$

denotes the infimum of all $a(E, X, \Delta)$, where $E$ runs through all divisors over $X$ whose center has nonempty intersection with $S$, except the irreducible components of $S$.

Using this notation, the above arguments established the following:

LEMMA 125. *Let $X$ be a normal variety and $S$ a reduced divisor on $X$. Let $D$ be an effective $\mathbb{Q}$-divisor that has no irreducible components in common with $S$. Assume that $K_X+S+D$ is $\mathbb{Q}$-Cartier and let $\bar{S}\to S$ denote the normalization.*

*Then for every divisor $E_S$ over $\bar{S}$ there is a divisor $E_X$ over $X$ such that*

$$a\big(E_X, X, S+D\big) = a\big(E_D, \bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big) \quad \textit{and} \quad \operatorname{center}_X E_X = \operatorname{center}_S E_D.$$

*In particular,*

$$\begin{array}{rcl} \operatorname{totaldiscrep}\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big) & \geq & \operatorname{discrep}\big(\operatorname{center}\subset S, X, S+D\big) \\ & \geq & \operatorname{totaldiscrep}(E\notin[S], \operatorname{center}\cap S\neq\emptyset, X, S+D). \end{array} \quad \square$$

Note that if $\operatorname{totaldiscrep}\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big) > -1$, then, by (72), $S$ is normal in codimension 1 and by (120), $S$ itself is normal. Thus, in these cases, we do not worry about the normalization and write $(S, \operatorname{Diff}_S(D))$ instead.

The above lemma (or similar results and conjectures) is frequently referred to as *adjunction* if we assume something about $X$ and obtain conclusions about $S$, or *inversion of adjunction* if we assume something about $S$ and obtain conclusions about $X$.

The main theorem in the subject (126) asserts that the inequalities in (125) are equalities. A special case of it was conjectured in [**Sho92**, 3.3] and extended to the current form in [$\mathbf{K^+92}$, 17.3]. The hard part is to establish the converse: we assume something about the lower dimensional pair $\big(S, \operatorname{Diff}_S(D)\big)$ and we want to conclude something about the higher dimensional pair $\big(X, S+D\big)$.

The surface case was proved in (87.5).

For applications the following two special cases are especially important.

(1) [$\mathbf{K^+92}$, 17.4] If $\operatorname{totaldiscrep}\big(S, \operatorname{Diff}_S(D)\big) > -1$ then $\operatorname{totaldiscrep}\big(E\notin[S], \operatorname{center}\cap S\neq\emptyset, X, S+D\big) > -1$.

(2) [**Kaw07**] If $\operatorname{totaldiscrep}\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big) \geq -1$ then $\operatorname{totaldiscrep}\big(\operatorname{center}\cap S\neq\emptyset, X, S+D\big) \geq -1$.

Both of these have been proved without using any form of the MMP and these have been important tools in higher dimensional birational geometry. The simplest proof of (1) is in [**KSC04**, Sec.6.4].

It was also observed in [$\mathbf{K^+92}$, 17.9–12] that (126) is implied by the full MMP for dlt pairs and this is essentially the proof that we present below. The only twist is that currently the MMP is known for klt pairs but not for dlt pairs.

THEOREM 126. *Notation and assumptions as in (125). Then*

$$\begin{array}{rcl} \operatorname{totaldiscrep}\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big) & = & \operatorname{discrep}\big(\operatorname{center}\subset S, X, S+D\big) \\ & = & \operatorname{totaldiscrep}(E\notin[S], \operatorname{center}\cap S\neq\emptyset, X, S+D). \end{array}$$

*In particular,*

(1) *If $\big(S, \operatorname{Diff}_S(D)\big)$ is canonical then $\big(X, S+D\big)$ is canonical near $S$. (In this case $S$ has to be disjoint from $D$.)*
(2) *If $\big(S, \operatorname{Diff}_S(D)\big)$ is klt then $\big(X, S+D\big)$ is plt in a neighborhood of $S$.*
(3) *If $\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big)$ is lc then $\big(X, S+D\big)$ is lc in a neighborhood of $S$.*

REMARK 127. The assumption that $K_X+S+D$ be $\mathbb{Q}$-Cartier is quite important. Let $C$ be a smooth projective curve. By taking a cone over $C\times\mathbb{P}^n$, we get a normal variety $X^{n+2}$ which contains a smooth divisor $S\cong\mathbb{A}^{n+1}$. There is only an isolated singular point at the vertex and, for $n \geq 1$, $\operatorname{Diff}_S(0)$ is the zero divisor. In particular, $\big(S, \operatorname{Diff}_S(0)\big)$ is even terminal. Nonetheless, if $g(C) \geq 2$ then $(X,S)$ is not lc. In fact, $(X,\Delta)$ is never lc, no matter how we choose a $\mathbb{Q}$-divisor $\Delta$.

As a first application of (126) we obtain that discrepancies behave well in flat families.

COROLLARY 128. *Let $X$ be normal and $D$ an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : X \to B$ be a flat, proper morphism to a smooth curve such that every irreducible component of $D$ dominates $B$. Set $X_b := f^{-1}(b)$ with normalization $\bar{X}_b \to X_b$. Then*

(1) $b \mapsto \operatorname{totaldiscrep}\big(\bar{X}_b, \operatorname{Diff}_{\bar{X}_b}(D|_{X_b})\big)$ *is lower semi continuous.*
(2) *The set* $\big\{b \in B : \big(\bar{X}_b, \operatorname{Diff}_{\bar{X}_b}(D|_{X_b})\big)$ *is klt (resp. lc)*$\big\}$ *is open in $B$.*
(3) *If $D = 0$ then* $\big\{b \in B : X_b$ *is canonical*$\big\}$ *is open in $B$.*

Proof. Fix $0 \in B$ and set $c = \operatorname{totaldiscrep}\big(X_0, \operatorname{Diff}_{\bar{X}_0}(D)\big)$. By (126), there is an open neighborhood $X_0 \subset U \subset X$ such that $\operatorname{totaldiscrep}(E \not\in [X_0], U, X_0 + D|_U) \geq c$.

Let $f : Y \to X$ be a log resolution of $(X, X_0 + D)$ and write $K_Y + \Delta \sim_{\mathbb{Q}} f^*(K_X + X_0 + D)$. For general $b \in B$, $X_b$ is contained in $U$ and, by Bertini's theorem, $Y_b := f^{-1}(X_b) \to X_b$ is a log resolution of $(X_b, D|_{X_b})$ and $K_{Y_b} + \Delta|_{Y_b} \sim_{\mathbb{Q}} f^*(K_{X_b} + D|_{X_b})$. Thus $\operatorname{totaldiscrep}\big(\bar{X}_b, \operatorname{Diff}_{\bar{X}_b}(D)\big) \geq c$, proving (1) and (2). Note that $(Y,0)$ is canonical iff $\operatorname{totaldiscrep}(Y,0) = 0$, hence (1) also implies (3). □

129 (Proof of (126)). Note that $\operatorname{totaldiscrep}\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big) \leq 0$ (since every divisor on $\bar{S}$ has discrepancy $\leq 0$) and $\operatorname{totaldiscrep}(E \not\in [S], X, S + D) \leq 0$ as shown by blowing up a divisor on $S$ along which $S$ and $X$ are smooth. Thus both sides in (126) are in $\{-\infty\} \cup [-1, 0]$.

By (125), $\operatorname{totaldiscrep}\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big) \geq \operatorname{totaldiscrep}(E \not\in [S], X, S + D)$, thus if $\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big)$ is not lc then $(X, S + D)$ is also not lc and both sides of (126) are $-\infty$. Similarly, if $\operatorname{totaldiscrep}(E \not\in [S], X, S + D) = 0$, that is, when $(X, S + D)$ is canonical, then $\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big)$ is also canonical.

Let $g : Y \to (X, S+D)$ be a log resolution with exceptional divisors $\{E_i : i \in I\}$ such that $g_*^{-1}(S + D)$ is smooth. By replacing $X$ with a suitable neighborhood of $S$, we may assume that $S \cap \operatorname{center}_X E_i \neq \emptyset$ for every $i$. Set

$$C_X := \operatorname{totaldiscrep}(E \not\in [S], \operatorname{center} \cap S \neq \emptyset, X, S + D).$$

By (124.3) we are done if there is an exceptional divisor, say $E_0$, such that $a(E_0, X, S + D) = C_X$ and $E_0 \cap f_*^{-1} S \neq \emptyset$. Unfortunately, the latter condition is hard to control.

Thus, our strategy is to find another $\mathbb{Q}$-factorial model $g : Z \to X$ and a $g$-exceptional divisor $E$ intersecting $S_Z := g_*^{-1} S$ such that $a(E, X, S + D) = C_X$. Write $K_Z + S_Z + \Delta \sim_{\mathbb{Q}} g^*(K_X + S + D)$. Since $Z$ is $\mathbb{Q}$-factorial, $E \cap S_Z = \cup F_i$ has pure codimension 1. By (87.5), each $F_i$ appears in $\operatorname{Diff}_{S_Z}(\Delta)$ with coefficient $\geq -a(E, X, S + D)$. Thus, by (124),

$$a\big(F_i, \bar{S}, \operatorname{Diff}_{\bar{S}}(D)\big) \leq a(E, X, S + D) = C_X,$$

as required.

In order to find such a $Z$, assume first that $(X, S + D)$ is lc and pick $0 \geq -c > C_X$ $(\geq -1)$ such that none of the exceptional divisors in a given log resolution $f : Y \to X$ have discrepancy in the half open interval $\big(C_X, -c\big]$. Now apply (40) with $c(E_i) = c$ for every $i$ to get $g : Z := X^{\mathrm{mc}} \to X$. By (40), the disrepancy of every

$g$-exceptional divisor $E_j \subset Z$ is $C_X$, and their union equals $\operatorname{Ex}(g)$. Furthermore, $g : Z \to X$ is not an isomorphism near $S$ since $\operatorname{discrep}(Z, g_*^{-1}(S+D) + c\sum E_j) \geq -c > C_X$. Thus at least one $g$-exceptional divisor $E_j$ intersects $g_*^{-1}S$.

The only thing that remains to show is that if $(X, S+D)$ is not lc then $\big(\bar S, \operatorname{Diff}_{\bar S}(D)\big)$ is also not lc. If we apply the previous method with $c = 1$, we get $g : Z \to X$ such that all $g$-exceptional divisors have disrepancy $\leq -1$. This only implies that $\big(\bar S, \operatorname{Diff}_{\bar S}(D)\big)$ is not klt. Thus we would need to use (40) for some $c > 1$, but this case does not hold in general. There are two ways to go around this problem.

129.1. *Special case.* Assume first that $S$ is $\mathbb{Q}$-Cartier. (This is the only case that we use in (128) and later in this book.)

Pick $0 < \eta \ll 1$ and apply (41) to $(X, (1-\eta)S + D)$ with $c = 1$ to get $g : Z_\eta \to X$ with $g$-exceptional divisors $E_i$. Note that as we replace $S$ by $(1-\eta)S$, we increase the discrepancy of every exceptional divisor whose center is contained in $S$. Thus we can choose $\eta$ such that no $g$-exceptional divisor with center in $S$ has discrepancy $-1$ and some have discrepancy $< -1$.

Since $S$ is $\mathbb{Q}$-Cartier, so is $g^*S$, hence if $g^*S \neq g_*^{-1}S$ then there is a $g$-exceptional divisor with center in $S$ that intersects $g_*^{-1}S$. We can complete the argument as before.

Othwerise, $g^*S = g_*^{-1}S$, hence $g_*^{-1}S$ intersects every $g$-exceptional divisor. By (41.5) there is at least one $g$-exceptional divisor whose discrepancy is $< -1$ and we are again done.

129.2. *General case.* The following method is due to Hacon. It can be viewed as one of the simplest instances of the subtle lifting techniques due to [**Siu98, BCHM06, Siu08**].

Let us start with a dlt model $f : Y \to X$ as in (41) with exceptional divisors $\{E_i : i \in I\}$. Fix an ample divisor $H$ on $Y$. By (39.2), for every $m \in \mathbb{N}$ there is a $\mathbb{Q}$-divisor $\Delta_{Y,m}$ such that $g_*^{-1}(S+D) + \sum E_i + \frac{1}{m}H \sim_{\mathbb{Q}} g_*^{-1}S + \Delta_{Y,m}$, $(Y, (1-\eta)g_*^{-1}S + \Delta_{Y,m})$ is klt for $\eta > 0$ and there is a $\mathbb{Q}$-factorial minimal model $f_m : Y_m \to X$.

Let $S_m$ (resp. $D_m, H_m, E_{im}, \Delta_m$) denote the birational transform of $S$ (resp. $D, H, E_i, \Delta_{Y,m}$) on $Y_m$. Set $\Sigma_m =: -\sum\big(a(E_i, X, S+D) + 1\big)E_{im}$. Thus $\Sigma_m$ is effective and its support consists of those divisors whose disrepancy is $< -1$. By construction, the following hold:

(3) $\big(Y_m, S_m + D_m + \sum E_{im} + \frac{1}{m}H_m\big)$ is lc and $\big(Y_m, \Delta_m\big)$ is klt.
(4) $K_{Y_m} + S_m + D_m + \sum E_{im} + \frac{1}{m}H_m \sim_{\mathbb{Q}, f_m} -\Sigma_m + \frac{1}{m}H_m$ and
(5) $D_m + \sum E_{im} + \frac{1}{m}H_m \sim_{\mathbb{Q}} \Delta_m$.

Our earlier arguments apply if $S_m$ meets $\Sigma_m$ for some $m$. We end the proof by deriving a contradiction if $S_m \cap \operatorname{Supp}\Sigma_m = \emptyset$ for every $m$.

Choose $m_0$ such that $m_0 a(E_i, X, S+D)$ are all integers. The exceptional divisors of $f_m$ are indexed by a subset of $I$. Thus there is an infinite subset $M \subset m_0\mathbb{N}$ such that the $f_m$ have the "same" exceptional divisors for $m \in M$. Thus, for $m \in M$, the models $Y_m$ are isomorphic to each other in codimension 1 and so the $\Sigma_m$ are birational transforms of each other. In particular, for any $a, b \in \mathbb{Z}$ the sheaves

$$(f_m)_*\mathcal{O}_{Y_m}\big(aH_m - \lfloor b\Sigma_m \rfloor\big) \quad \text{are independent of } m \in M. \tag{129.6}$$

Next we compute the composite

$$\psi_m : (f_m)_*\mathcal{O}_{Y_m}\big(H_m - m\Sigma_m\big) \to (f_m)_*\mathcal{O}_{Y_m}\big(H_m\big) \to (f_m)_*\mathcal{O}_{S_m}\big(H_m|_{S_m}\big)$$

in two different ways.

First fix $n$ and let $m$ increase. The shaves $(f_n)_*\mathcal{O}_{Y_n}\big(H_n - m\Sigma_n\big)$ form a decreasing sequence of subsheaves of $(f_n)_*\mathcal{O}_{Y_n}\big(H_n\big)$. Furthermore, if $\mathcal{I}$ denotes the ideal sheaf of $f_n(\Sigma_n)$ (which is independent of $n \in M$), then

$$(f_n)_*\mathcal{O}_{Y_n}\big(H_n - m\Sigma_n\big) \subset \mathcal{I} \cdot (f_n)_*\mathcal{O}_{Y_n}\big(H_n\big) \quad \text{for } m \gg n.$$

By (129.6), $(f_n)_*\mathcal{O}_{Y_n}\big(H_n - m\Sigma_n\big) = (f_m)_*\mathcal{O}_{Y_m}\big(H_m - m\Sigma_m\big)$, thus $\psi_m$ is not surjective for $m \gg 1$ since $S \cap f_n(\Sigma_n) \neq \emptyset$.

If $S_m \cap \operatorname{Supp}\Sigma_m = \emptyset$ then $\mathcal{O}_{Y_m}\big(H_m - m\Sigma_m\big)\big|_{S_m} = \mathcal{O}_{S_m}\big(H_m|_{S_m}\big)$, hence we have an exact sequence

$$0 \to \mathcal{O}_{Y_m}\big(H_m - m\Sigma_m - S_m\big) \to \mathcal{O}_{Y_m}\big(H_m - m\Sigma_m\big) \to \mathcal{O}_{S_m}(H_m) \to 0.$$

By pushing it forward we get

$$(f_m)_*\mathcal{O}_{Y_m}\big(H_m - m\Sigma_m\big) \xrightarrow{\psi_m} (f_m)_*\mathcal{O}_{S_m}\big(H_m|_{S_m}\big) \to R^1(f_m)_*\mathcal{O}_{Y_m}\big(H_m - m\Sigma_m - S_m\big).$$

Using (3–5),

$$H_m - m\Sigma_m - S_m \sim_{\mathbb{Q}} K_{Y_m} + D_m + \textstyle\sum E_{im} + \frac{1}{m}H_m + (m-1)\big(-\Sigma_m + \frac{1}{m}H_m\big),$$

$D_m + \sum E_{im} + \frac{1}{m}H_m \sim_{\mathbb{Q}} \Delta_m$, $(Y_m, \Delta_m)$ is klt and $-\Sigma_m + \frac{1}{m}H_m$ is $f_m$-nef. Hence, by the vanishing theorem [**KM98**, 2.68], $R^1(f_m)_*\mathcal{O}_{Y_m}\big(H_m - m\Sigma_m - S_m\big) = 0$. Thus $\psi_m$ is surjective, a contradiction. $\square$

## 6. Log canonical centers

For a lc pair $(X, \Delta)$ it is especially interesting and useful to study exceptional divisors with discrepancy $-1$. These divisors, and their centers on $X$ play a crucial role in the inductive treatment of lc pairs.

DEFINITION 130. Let $(X, \Delta)$ be a lc pair with $\Delta$ not necessarily effective. We say that an irreducible subvariety $Z \subset X$ is a *log canonical center* or *lc center* of $(X, \Delta)$ if there is a divisor $E$ over $X$ such that $a(E, X, \Delta) = -1$ and $\operatorname{center}_X E = Z$.

Let $f : Y \to X$ be a proper birational morphism and write $K_Y + \Delta_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta)$ as in (7). Then, by (12), the lc centers of $(X, \Delta)$ are exactly the images of the lc centers of $(Y, \Delta_Y)$.

If $X$ is smooth and $\Delta$ is snc, then the lc centers of $(X, \Delta)$ are exactly the irreducible components of the various intersections $D_{i_1} \cap \cdots \cap D_{i_s}$ where the $D_k$ appear in $\Delta$ with coefficient 1. This follows easily from (15), or see (132) for a more general case.

In particular, there are only finitely many lc centers. Their union is called the *non-klt locus* of $(X, \Delta)$. It is denoted by $\operatorname{nklt}(X, \Delta)$. (Some authors call this the "log canonical locus," but this violates standard usage.)

131. Let $X$ be a normal variety and $\Delta$ a $\mathbb{Q}$-divisor on $X$ such that $(X, \Delta)$ is $\mathbb{Q}$-Cartier. The set of points where $X$ is smooth and $\Delta$ is a snc divisor is open, let us call it the *snc locus* of $(X, \Delta)$ and denote it by $\operatorname{snc}(X, \Delta)$. Its complement is the *non-snc locus*, denoted by $\operatorname{non-snc}(X, \Delta)$ (**??**).

Recall that $(X, \Delta)$ is called *dlt* if $\Delta$ is effective and $a(E, X, \Delta) > -1$ for every divisor $E$ whose center on $X$ is contained in non-snc$(X, \Delta)$.

Write $\Delta = D + \Delta_{<1}$ where $D = D_1 + \cdots + D_r = \lfloor \Delta \rfloor$ are the divisors that appear in $\Delta$ with coefficient $= 1$ and $\Delta_{<1} = \Delta - \lfloor \Delta \rfloor$ are the divisors that appear in $\Delta$ with coefficient $< 1$.

**Lc centers in the dlt case.**

THEOREM 132. [**?**, Sec.3.9] *Let $(X, \Delta)$ be a dlt pair and $D_1, \dots, D_r$ the irreducible divisors that appear in $\Delta$ with coefficient 1.*

(1) *The $s$-codimensional lc centers of $(X, \Delta)$ are excatly the irreducible components of the various intersections $D_{i_1} \cap \cdots \cap D_{i_s}$ for some $\{i_1, \dots, i_s\} \subset \{1, \dots, r\}$.*
(2) *Every irreducible component of $D_{i_1} \cap \cdots \cap D_{i_s}$ is normal and of pure codimension $s$.*

Proof. Let $E$ be a divisor over $X$ such that $a(E, X, \Delta) = -1$ and $Z = \operatorname{center}_X E$. By localizing at the generic point of $Z$, we may assume that $Z$ is a closed point of $X$. By the dlt assumption, $X$ is smooth at $Z$ and $\Delta$ is snc. If $\dim_Z X = n$, there are $n$ snc divisors $B_1, \dots, B_n$ through $Z$ such that $\Delta = \sum a_i B_i$ for some $0 \le a_i \le 1$ (where we can ignore the components of $\Delta$ that do not pass through $Z$). Set $\Delta' := \sum B_i$. Then $a(E, X, \Delta') \ge -1$ by (9) and $a(E, X, \Delta) > a(E, X, \Delta') \ge -1$ by (10) unless $a_i = 1$ for every $i$. Thus every $B_i$ appears in $\Delta$ with coefficient $= 1$ - hence is one of the $D_j$ - and $Z$ is an irreducible component of the intersection of the corresponding $D_j$.

We use induction on $s$ to prove the rest. We may assume that $i_j = j$. By (72) each $D_i$ is smooth in codimension 1 and $S_2$ (even CM) by (120). Thus each $D_i$ is normal. By (104), the support of $\mathcal{O}_{D_i} + \mathcal{O}_{D_j} / \mathcal{O}_{D_i + D_j}$ has pure codimension 2 in $X$, hence $D_i \cap D_j$ has pure codimension 2 in $X$.

By (133) $\bigl(D_s, \operatorname{Diff}_{D_s}(\Delta - D_s)\bigr)$ is dlt. Thus, by induction, every irreducible component of $D_1 \cap \cdots \cap D_s$ is normal, has pure codimension $s$ and is an lc center of $\bigl(D_s, \operatorname{Diff}_{D_s}(\Delta - D_s)\bigr)$. By (125), these are all lc centers of $(X, \Delta)$ as well. $\square$

LEMMA 133. *If $(X, D + \Delta)$ is dlt and $D$ is irreducible then $(D, \operatorname{Diff}_D \Delta)$ is also dlt.*

Proof. We saw at the beginning of the proof of (132.2) that $D$ is normal.

Let $Z \subset D$ be an lc center of $(D, \operatorname{Diff}_D \Delta)$. Then there is a divisor $E_D$ over $D$ whose center is $Z$ such that $a(E_D, D, \operatorname{Diff}_D \Delta) = -1$. Thus, by (125) there is a divisor $E_X$ over $X$ whose center is $Z$ such that $a(E_X, X, D + \Delta) = -1$. Since $(X, D + \Delta)$ is dlt this implies that $(X, D + \Delta)$ is snc at the generic point of $Z$. Thus $\operatorname{Diff}_D \Delta = \Delta|_D$ and $(D, \operatorname{Diff}_D \Delta)$ is snc at the generic point of $Z$.

The converse fails, for instance for $\bigl(\mathbb{A}^3, (z = 0) + (xy = z^m)\bigr)$. $\square$

134 (Higher codimension Poincaré residue maps). First, let $X$ be a smooth variety and $D_1 + \cdots + D_r$ a snc divisor on $X$. We can iterate the codimension 1 Poincaré residue maps (121) to obtain higher codimension Poincaré residue maps

$$\mathcal{R}_{X \to D_1 \cap \cdots \cap D_r} : \omega_X(D_1 + \cdots + D_r)|_{D_1 \cap \cdots \cap D_r} \cong \omega_{D_1 \cap \cdots \cap D_r} \tag{134.1}$$

defined by

$$\mathcal{R}_{X \to D_1 \cap \cdots \cap D_r} := \mathcal{R}_{D_1 \cap \cdots \cap D_{r-1} \to D_1 \cap \cdots \cap D_r} \circ \cdots \circ \mathcal{R}_{D_1 \to D_1 \cap D_2} \circ \mathcal{R}_{X \to D_1}.$$

Note that these maps are, however, defined only up to sign. It is enough to check this for 2 divisors. As a local model, take $X := \mathbb{A}^n$ and $D_i := (x_i = 0)$. For any $1 \le r \le n$, a generator of $\omega_X(D_1 + \cdots + D_r)$ is given by

$$\sigma := \frac{1}{x_1 \cdots x_r} dx_1 \wedge \cdots \wedge dx_n.$$

If we restrict first to $D_1$ and then to $D_2$, we get

$$\mathcal{R}_{X \to D_1 \cap D_2}(\sigma) = \frac{1}{x_3 \cdots x_r} dx_3 \wedge \cdots \wedge dx_n.$$

If we restrict first to $D_2$ and then to $D_1$, then we need to interchange $dx_1$ and $dx_2$ first, hence we get

$$\mathcal{R}_{X \to D_1 \cap D_2}(\sigma) = \frac{-1}{x_3 \cdots x_r} dx_3 \wedge \cdots \wedge dx_n.$$

Note also that, if $W = D_1 \cap \cdots \cap D_r$ and $Z = W \cap D_{r+1} \cap \cdots \cap D_s$ then, by construction,

$$\mathcal{R}_{X \to W} = \pm \mathcal{R}_{W \to Z} \circ \mathcal{R}_{X \to W} \tag{134.2}$$

Assume next that $X$ is a normal variety, $D_1, \dots, D_r \subset X$ integral divisors and $\Delta$ a $\mathbb{Q}$-divisor such that $K_X + \sum D_i + \Delta$ is $\mathbb{Q}$-Cartier. Let $Z^0 \subset \operatorname{snc}(X, \sum D_i) \cap D_1 \cap \cdots \cap D_r$ be an open subset and $\pi_Z : Z \to X$ the normalization of its closure. Assume that $Z^0$ is disjoint from $\operatorname{Supp} \Delta$.

Choose $m > 0$ even such that $m(K_X + \sum D_i + \Delta)$ is Cartier. We can apply (134.1) to the snc locus of $(X, D_1 + \cdots + D_r)$ to obtain an isomorphism

$$\mathcal{R}^m_{X \to Z^0} : \omega_X^{[m]}(m\textstyle\sum D_i + m\Delta)|_{Z^0} \xrightarrow{\cong} \omega_{Z^0}^{[m]}. \tag{134.3}$$

As in (122), this gives a rational section of

$$\operatorname{Hom}_Z\Bigl(\pi_Z^* \omega_X^{[m]}(m\textstyle\sum D_i + m\Delta), \omega_Z^{[m]}\Bigr)$$

There is thus a unique $\mathbb{Q}$-divisor $\operatorname{Diff}^*_Z \Delta$, called the *different* such that (134.3) extends to an isomorphism

$$\mathcal{R}^m_{X \to Z} : \omega_X^{[m]}(m\textstyle\sum D_i + m\Delta)|_Z \xrightarrow{\cong} \omega_Z^{[m]}(m \cdot \operatorname{Diff}^*_Z \Delta). \tag{134.4}$$

Note that this formula simultaneously defines $\mathcal{R}^m_{X \to Z}$ and $\operatorname{Diff}^*_Z \Delta$ and that $\mathcal{R}^m_{X \to Z}$ is defined only for those values of $m$ for which $m(K_X + \sum D_i + \Delta)$ is Cartier.

*Remark on the notation.* In (87) the orginal $\operatorname{Diff}_D$ is set up so that if $\Delta = D + D'$ then the restriction of $K_X + \Delta = K_X + D + D'$ to $D$ is $K_D + \operatorname{Diff}_D D' = K_D + \operatorname{Diff}_D(\Delta - D)$. That is, we first remove $D$ from $\Delta$ and then take the different. For higher codimension lc centers $Z \subset X$, it does not make sense to "first remove $Z$ from $\Delta$." Thus we need the new notation $\operatorname{Diff}^*$. In the classical case, $\operatorname{Diff}^*_D \Delta = \operatorname{Diff}_D(\Delta - D)$.

Applying this to the dlt case, we obtain the following

PROPOSITION 135. *Let $(X, \Delta)$ be dlt and $Z \subset X$ an lc center. Then $\operatorname{Diff}^*_Z \Delta$ is an effective $\mathbb{Q}$-divisor such that*

(1) *$(Z, \operatorname{Diff}^*_Z \Delta)$ is dlt,*
(2) *for $m > 0$ even such that $m(K_X + \Delta)$ is Cartier, the Poincaré residue map gives an isomorphism*

$$\mathcal{R}^m_{X \to Z} : \omega_X^{[m]}(m\Delta)|_Z \xrightarrow{\cong} \omega_Z^{[m]}(m \cdot \operatorname{Diff}^*_Z \Delta).$$

(3) *If $W \subset Z$ is another lc center of $(X, \Delta)$ then $W$ is also an lc center of $(Z, \operatorname{Diff}^*_Z \Delta)$ and*

$$\operatorname{Diff}^*_W \Delta = \operatorname{Diff}^*_W\big(\operatorname{Diff}^*_Z \Delta\big).$$

Proof. By the definition, (134.4) and (134.2), (3) holds if $Z$ is a divisor on $X$ and $W$ a divisor on $Z$. Thus, by induction, (3) also holds whenever $Z$ and $W$ are irreducible components of complete intersections of divisors in $\lfloor \Delta \rfloor$. By (132) this covers all cases. Now (3) and induction gives (1) and (2) was part of the definition of $\operatorname{Diff}^*_Z \Delta$. □

**$\mathbb{P}^1$-linked lc centers.**

One of the difficulties of dealing with lc centers in the non-dlt case is that the Poincaré residue map is more complicated. A possible solution found in [**Kaw98, Kol07b**] is to define a $\mathbb{Q}$-linear equivalence class $J_Z$ such that

$$\mathcal{R}^m_{X \to Z} : \omega_X^{[m]}(m\Delta)|_Z \xrightarrow{\cong} \omega_Z^{[m]}(mJ_Z + m \operatorname{Diff}^*_Z \Delta).$$

(Note that $\operatorname{Diff}^*_Z \Delta$ is an actual divisor, whereas $J_Z$ is only a $\mathbb{Q}$-linear equivalence class. There does not seem to be any sensible way to pick a divisor corresponding to $J_Z$.) With this formulation, the two sides are not symmetric. It should be possible to work out a symmetric variant, but this seems to involve some technical issues about variations of mixed Hodge structures.

Here we present another approach to these problems. Instead of working directly on an lc center $Z \subset X$, we consider a dlt model $f : (Y, \Delta_Y) \to (X, \Delta)$ and a minimal lc center $Z_Y \subset Y$ that dominates $Z$. We then try to study $Z$ through the induced map $f|_{Z_Y} : Z_Y \to Z$.

In general $Z_Y$ is not unique, but, as we see next, the different $Z_Y$ are closely related to each other.

Definition 136. Let $(X, \Delta)$ be a dlt pair and $g : X \to S$ a proper morphism. Let $Z_1, Z_2 \subset X$ be two lc centers. We say that $Z_1, Z_2$ are directly $\mathbb{P}^1$-linked if there is an lc center $W \subset X$ containing the $Z_i$ such that

(1) $g(W) = g(Z_1) = g(Z_2)$ and
(2) there is a variety $V$ and a birational map

$$\phi : \big(Z_1, Z_2, W\big) \dashrightarrow \big(0, \infty, \mathbb{P}^1\big) \times V$$

such that $\phi$ maps $Z_1$ (resp. $Z_2$) birationally to $\{0\} \times V$ (resp. $\{\infty\} \times V$).

We say that $Z_1, Z_2 \subset X$ are $\mathbb{P}^1$-*linked* if there is a sequence of lc centers $Z'_1, \dots, Z'_m$ such that $Z'_1 = Z_1, Z'_m = Z_2$ and $Z'_i$ is directly $\mathbb{P}^1$-linked to $Z'_{i+1}$ for $i = 1, \dots, m-1$ (or $Z_1 = Z_2$).

The following theorem plays an important role in the study of lc centers in the non-dlt case.

Theorem 137. *Let $(X, \Delta)$ be dlt and $f : X \to S$ a proper morphism such that $K_X + \Delta \sim_{\mathbb{Q},f} 0$. Let $s \in S$ be a point such that $f^{-1}(s)$ is connected (as a $k(s)$-scheme). Let $Z \subset X$ be minimal (with respect to inclusion) among the lc centers of $(X, \Delta)$ such that $s \in f(Z)$. Let $W \subset X$ be an lc center of $(X, \Delta)$ such that $s \in f(W)$.*

*Then there is an lc center $Z_W \subset W$ such that $Z$ and $Z_W$ are $\mathbb{P}^1$-linked.*

*In particular, if $Z_i \subset X : i = 1, 2$ are minimal among the lc centers of $(X, \Delta)$ such that $s \in f(Z_i)$, then $f(Z_1) = f(Z_2)$ and the $Z_i$ are $\mathbb{P}^1$-linked.*

Note that we do not assume that $f$ has connected fibers; this is useful over non-closed fields. The following example illustrates some of the subtler aspects.

EXAMPLE 138. Set $X = \mathbb{A}^3$ and $D_1, D_2, D_3$ planes intersecting only at the origin. Let $\pi : B_0X \to X$ denote the blow-up of the origin with exceptional divisor $E$. Then $K_{B_0X} + E + \sum D'_i \sim \pi^*(K_X + \sum D_i)$ where $D'_i := \pi_*^{-1} D_i$. There are 3 minimal lc centers over 0, given by $p_i := E \cap D'_{i-1} \cap D'_{i+1}$ (with indexing modulo 3).

Assume now that we are over $\mathbb{Q}$, $D_1$ is defined over $\mathbb{Q}$ and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ interchanges $D_2, D_3$. Now there are 2 minimal lc centers over 0. One is $p_1$ the other is the irreducible $\mathbb{Q}$-scheme $p_2 + p_3$. Thus $p_1$ and $p_2 + p_3$ can not be $\mathbb{P}^1$-linked. This is not a contradiction since $(B_0X, E + \sum D'_i)$ is not dlt; the divisor $D'_2 + D'_3$ (which is irreducible over $\mathbb{Q}$) is not normal. We get a dlt model by blowing up the curve $D'_2 \cap D'_3$. Now there are 2 minimal lc centers over 0, both isomorphic to $p_2 + p_3$.

Similarly, if $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ permutes the 3 planes, then we need to blow up all 3 intersections $D'_i \cap D'_j$ to get a dlt model. Over $\bar{\mathbb{Q}}$, there are 6 minimal lc centers over 0. Over $\mathbb{Q}$ there is either only one (if $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the planes as the symmetric group $S_3$) or two, both consisting of 3 conjugate points and isomorphic as $\mathbb{Q}$-schemes to each other (if $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ permutes cyclically).

Proof. We use induction on $\dim X$.

Write $\lfloor \Delta \rfloor = \sum D_i$. By passing to a strict étale neighborhood of $s \in S$ we may assume that each $D_i \to Y$ has connected fiber over $s$ and every lc center of $(X, \Delta)$ intersects $f^{-1}(s)$. (We need a *strict* étale neighborhood, that is, the residue field at $s$ is unchanged, to make sure that $f^{-1}(s)$ stays connected.)

Assume first that $f^{-1}(s) \cap \sum D_i$ is connected. By suitable indexing, we may assume that $Z \subset D_1$, $W \subset D_r$.

By induction, we can apply (137) to $D_1 \to S$ with $Z$ as $Z$ and $D_1 \cap D_2$ as $W$. We get that there is an lc center $Z_2 \subset W$ such that $Z$ and $Z_2$ are $\mathbb{P}^1$-linked. Note that $Z_2$ is an lc center of $\bigl(D_1, \mathrm{Diff}^*_{D_1}(\Delta)\bigr)$. By adjunction, it is also an lc center of $(X, \Delta)$ hence, by (126), an lc center of $\bigl(D_2, \mathrm{Diff}^*_{D_2}(\Delta)\bigr)$.

Next we apply (137) to $D_2 \to S$ with $Z_2$ as $Z$ and $D_2 \cap D_3$ as $W$, and so on. At the end we work on $D_r \to S$ with $Z_r$ as $Z$ and $W$ as $W$ to get an lc center $Z_W \subset W$ such that $Z$ and $Z_W$ are $\mathbb{P}^1$-linked. This proves the first claim if $f^{-1}(s) \cap \sum D_i$ is connected.

If $f^{-1}(s) \cap \sum D_i$ is disconnected, then by (141) $Z$ and $W$ are the only lc centers and there is a variety $V$ and a birational map $\phi : X \dashrightarrow \mathbb{P}^1 \times V$ such that $\phi$ maps $Z$ (resp. $W$) birationally to $\{0\} \times V$ (resp. $\{\infty\} \times V$). Thus $Z$ and $W$ are $\mathbb{P}^1$-linked.

Assume finally that $Z_1, Z_2 \subset X$ are minimal among the lc centers of $(X, \Delta)$ such that $s \in f(Z_i)$. By the first part, $Z_2$ contains an lc center $Z'_2$ that is $\mathbb{P}^1$-linked to $Z_1$. Since $Z_2$ is minimal, $Z'_2 = Z_2$ and hence $Z_1$ and $Z_2$ are $\mathbb{P}^1$-linked. This implies that $f(Z_1) = f(Z_2)$. $\square$

The following lemma can be viewed as a common generalization of the connectedness theorems [**K$^+$92**, 17.4] and [**Kol07a**, 9].

LEMMA 139. *Let $X$ be a smooth variety, $S$ a normal variety and $f : X \to S$ a proper morphism with connected fibers. Write $K_X \sim_{f,\mathbb{Q}} A - B - \Delta$ where*

(1) *$A + B + \Delta$ is an snc divisor,*
(2) *$A$ is an effective $\mathbb{Z}$-divisor, $B$ is a reduced $\mathbb{Z}$-divisor,*

(3) *$B$ has no common irreducible components with $A+\Delta$,*
(4) *$\lfloor\Delta\rfloor = 0$ and*
(5) *$f_*\mathcal{O}_X(A)$ is an invertible sheaf on $S$.*

*Then, for every $s\in S$,*

(6) *either every lc center of $(X,B)$ that intersects $f^{-1}(s)$ also dominates $S$,*
(7) *or $f^{-1}(s)\cap\operatorname{Supp}B$ is connected. (This is always the case if $f$ is birational.)*

Proof. As before we may assume that every lc center of $(X,B)$ intersects $f^{-1}(s)$. Write $B=B_h+B_v$ where every irreducible component of $B_h$ dominates $S$ and $B_v$ does not dominate $S$. By blowing up various lc centers of $(X,B_h)$ if necessary, we may assume that every lc center of $(X,B_h)$ dominates $S$.

Choose any $B_0\subset B_h$, write $B=B_0+B_0'$ and let $\sum_{i\ge 1}B_i$ be the connected components of $B_0'$. Consider the exact sequence

$$0\to\mathcal{O}_X(A-B_0')\to\mathcal{O}_X(A)\to\textstyle\sum_{i\ge 1}\mathcal{O}_{B_i}\big(A|_{B_i}\big)\to 0$$

and its push-forward

$$\mathcal{O}_S\cong f_*\mathcal{O}_X(A)\xrightarrow{r}\textstyle\sum_{i\ge 1}f_*\mathcal{O}_{B_i}\big(A|_{B_i}\big)\xrightarrow{\delta}R^1f_*\mathcal{O}_X(A-B_0').$$

139.8 *Claim.* If $B_1$ does no dominate $S$ then $B_0'=B_1$.

Proof. Since $A-B_0'\sim_{f,\mathbb{Q}}K_X+\Delta+B_0$, we see that $R^1f_*\mathcal{O}_X(A-B_0')$ is torsion free by (**??**). On the other hand, $f_*\mathcal{O}_{B_1}\big(A|_{B_1}\big)$ is a nonzero torsion sheaf over $\mathcal{O}_S$ which is therefore in $\ker\delta$. Thus the image of $r$ contains $f_*\mathcal{O}_{B_1}\big(A|_{B_1}\big)$, hence it can be written as

$$\operatorname{im}r=f_*\mathcal{O}_{B_1}\big(A|_{B_1}\big)+M\quad\text{for some}\quad M\subset\textstyle\sum_{i\ge 2}f_*\mathcal{O}_{B_i}\big(A|_{B_i}\big).$$

Since $\operatorname{im}r$ is a cyclic $\mathcal{O}_{s,S}$-sheaf, this implies that $M=0$ (after possibly shrinking $S$). Thus the maps $f_*\mathcal{O}_X(A)\to f_*\mathcal{O}_{B_i}\big(A|_{B_i}\big)$ are zero for $i\ge 2$.

On the other hand, the constant section of $\mathcal{O}_X(A)$ restricts to a nonzero section of $\mathcal{O}_{B_i}\big(A|_{B_i}\big)$, hence the maps $f_*\mathcal{O}_X(A)\to f_*\mathcal{O}_{B_i}\big(A|_{B_i}\big)$ are all nonzero. □

Assume now that $f^{-1}(s)\cap\operatorname{Supp}B$ is disconnected and not every lc center dominates $S$. In necessary, we can blow up such an lc center to reduce to the case when an irreducible component of $B$ does not dominate $S$. By passing to an étale neighborhood of $s\in S$, we may assume that $\operatorname{Supp}B$ is disconnected. Write $B=D_1+D_2$ where the $D_i$ are disjoint and $D_1$ contains a divisor that does not dominate $S$. Apply (139.8) with $B_0=(D_1)_h$ and $B_1=(D_1)_v$ to conclude that $D_2=0$, a contradiction. □

Corollary 140. *Let $g:Y\to S$ be a proper morphism with connected fibers between normal varieties. Assume that $(Y,\Delta)$ is lc and $K_Y+\Delta\sim_{\mathbb{Q},g}0$. Then, for any $s\in S$, $g^{-1}(s)\cap\operatorname{nklt}(Y,\Delta)$ is*

(1) *either connected,*
(2) *or every lc center of $(Y,\Delta)$ that intersects $g^{-1}(s)$ also dominates $S$.*

Proof. Let $\pi:X\to Y$ be a log resolution of $(Y,\Delta)$. Write $\pi^*(K_Y+\Delta)\sim_{\mathbb{Q}}K_X-A+B+\Delta_X$ and note that the assumptions of (139) are satisfied by $f:=g\circ\pi$. Note that

$$g^{-1}(s)\cap\operatorname{nklt}(Y,\Delta)=\pi\Big(f^{-1}(s)\cap\operatorname{Supp}B\Big).$$

Thus either $f^{-1}(s) \cap B$ and hence $g^{-1}(s) \cap \mathrm{nklt}(Y, \Delta)$ are both connected, or every lc center of $(X, B)$ (equivalently, of $(X, B + \Delta_X - A)$) that intersects $f^{-1}(s)$ also dominates $S$. By (130) the lc centers of $(Y, \Delta)$ are exactly the images of the lc centers of $(X, B + \Delta_X - A)$; and this gives (2). $\square$

Up to birational equivalence, there is only one way that the case (140.2) can happen.

LEMMA 141. *(cf. [**K$^+$92**, 12.3.1]) Let $(X, \Delta)$ be a dlt pair defined over a field $k$ such that $E := \lfloor \Delta \rfloor$ is disconnected. Then there is a $k$-variety $V$ and a birational map $\phi : X \dashrightarrow \mathbb{P}^1 \times V$ such that $\phi$ maps $E$ birationally to $\{0, \infty\} \times V$.*

Proof. Write $\Delta = E + \Delta'$ and run the $(X, \Delta')$-MMP. This is possible by [**BCHM06**, 1.3.2].

Every step is numerically $K_X + E + \Delta'$-trivial, hence $E$ is ample on every extremal ray. Therefore a connected component of $E$ can never be contracted by a birational contraction. By (139) $E$ stays disconnected. At some point, we must encounter a Fano-contraction $p : (X^*, \Delta^*) \to V$ where $E^*$ is ample on the general fiber. By (140) every irreducible component of $E^*$ dominates $S$.

Since the relative Picard number is one, every irreducible component of $E^*$ is relatively ample. Thus a general fiber of $p$ contains at least 2 disjoint ample divisors. This is only possible if $p$ has fiber dimension 1, the generic fiber is a smooth rational curve and $E^*$ has 2 irreducible components which are sections of $p$. Thus $X^*$ is birational to $V \times \mathbb{P}^1$. (For a more precise description of $X^* \to V$ see the proof of [**KK09**, 5.1].) $\square$

Finally let us see what happens with lc centers for birational maps between dlt pairs.

LEMMA 142. *Let $(X, \Delta)$ be dlt and $f : (Y, \Delta_Y) \to (X, \Delta)$ a log resolution. Let $T \subset X$ be an lc center of $(X, \Delta)$ and $S \subset Y$ a minimal lc center of $(Y, \Delta_Y)$ dominating $T$. Then $f : S \to T$ is birational.*

Proof. It is enough to prove that $Y \to X$ is dominated by a $Y' \to X$ that satisfies the conclusion. We construct such $Y' \to X$ in 2 steps.

As we see in (**??**.2), there is a log resolution $g : X' \to X$ of $(X, \Delta)$ such that every lc center of $(X', \Delta')$ maps birationally to an lc center of $(X, \Delta)$. This reduces us to the case when $\Delta$ is a snc divisor. Then, by induction, it is sufficient to show that (142) holds for one blow up $p_Z : B_Z X \to X$ where $Z \subset X$ has snc with $\Delta$.

If $T \not\subset Z$ then $p_Z$ is an isomorphism over the generic point of $T$ and the birational trasnform of $T$ is the only lc center of $\big(B_Z X, \Delta_{B_Z X}\big)$ that dominates $T$.

It remains to consider the case when $T \subset Z$. Let $D_1, \dots, D_r$ be the components of $\lfloor \Delta \rfloor$ such that $T$ is an irreducible component of $D_1 \cap \cdots \cap D_r$. Since $Z$ has snc with $\operatorname{Supp} \Delta$, we see that $Z$ is the intersection of some of the $D_i$, say $Z = D_1 \cap \cdots \cap D_j$.

The exceptional divisor $E_Z$ of $p_Z$ is a $\mathbb{P}^{j-1}$-bundle over $Z$ and it appears in $\Delta_{B_Z X}$ with coefficient 1. There are $r$ different minimal lc centers that dominate $Z$, obtained by intersecting $E_Z$ with $j-1$ of the birational transforms of the $D_1, \dots, D_j$ and with the birational transforms of the $D_{j+1}, \dots, D_r$. Each of these centers maps isomorphically to $Z$. (Note that this step could fail if $\Delta$ is only nc but not snc.) $\square$

## 7. Du Bois, by SK

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